10. "On the solution of linear equations," by Dr. Aubrey J. Kempner, University of Colorado.

11. "Some problems in elementary research," by Dr. J. L. Gibson, University of Utah, by invitation.

Abstracts of papers follow:

1. Professor Clark showed how an assumption of continuity in the variation of soil fertility would offset the unreliability of results in experimentation plots due to their insufficient number. Using adjoining plots, a correlation is effected which uses a sufficient number of plots to overcome the unreliability of first results.

2. Professor Hutchinson gave a brief outline of the subject of cartography, indicating some of the problems of mathematical interest in that field.

3. This is an outline of the methods used by Felix Klein in developing the icosahedral equation and showing its use in the solution of the quintic equation.

4. Mr. Rote showed a construction for a square which approximated closely the area of a circle.

5. This paper deals with the problem of finding those properties of the hyper-surface of bi-secants to an algebraic curve in four dimensions that can be determined by projecting the given curve from a given line onto a plane.

6. Mr. McNatt developed in a new way the known formula for the distance between the in-center and circum-center of a triangle.

8. Professor Mallory emphasized the fact that the present tendency in the selection of subject matter of high school mathematics was toward the adaptation of material to pupils' ability and toward a more informal treatment of the subject generally.

9. Professor Nelson explained briefly a device consisting of nine cards which may be used to determine the age of a person. When the cards are placed according to instructions the age of the person appears in large type through an opening in the back of the pile of cards.

10. This paper was published in the August-September issue of this "Monthly."

11. The general parametric equations of the space and body centrodes of certain disks and disk-like bodies, and other surfaces and solids supported by and rolling between two intersecting planes are found. The abscissas of the instantaneous axes of rotation give the values of definite integrals whose peculiarities have in these problems specific physical meanings. This makes it possible to use the planes as mechanical analyzers of many integrals, including some elliptic and hyperelliptic integrals. Points rigidly attached to the rolling bodies generate roulettes, the abscissa of each point of which, using the general equations, contains a definite integral. These curves, under certain conditions, degenerate into many well known forms, such as the cycloids. If we assume the equations of the space and body centrodes and study the integrals and equations of curves which follow from them, we find a field in which research has been done by students of limited mathematical attainments. Other problems of

a mechanical nature leading either to new methods or new results were mentioned. It was suggested that more attention be paid to the finding of this type of problem for the purpose of stimulating research earlier in the case of students specializing in mathematics.

The members and friends of the association were guests of the State Teachers College on the evening of April 12 at a banquet. President Finley acted as toastmaster. The address of welcome was given by President Frazier of State Teachers College and the response was given by Professor G. W. Gorrell of Denver University, after which there was a very interesting talk by Dean J. L. Gibson of the University of Utah. Dean Gibson recounted his experiences in visiting Germany and German mathematicians after the war.

A. J. LEWIS, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The regular spring meeting of the Minnesota Section was held at the College of St. Catherine, St. Paul, Minnesota, on Saturday, May 11, 1929. At the request of the chairman, Sister Alice Irene, Professor Dunham Jackson presided at the morning and afternoon sessions.

The attendance was 60 at the luncheon and 80 at the regular session, and included the following 30 members of the Association: W. O. Beal, R. W. Brink, W. E. Brooke, W. H. Bussey, Elizabeth Carlson, H. H. Dalaker, J. M. Earl, Margaret Eide, Gladys Gibbens, C. H. Gingrich, S. Guttman, D. Jackson, C. M. Jensen, W. H. Kirchner, E. L. Mickelson, Marie Ness, M. A. Nordgaard, G. C. Priester, Inez Rundstrom, R. E. Scammon, R. R. Shumway, Sister Alice Irene, Sister Prudentia Morin, F. J. Taylor, Ella Thorp, A. L. Underhill, M. B. White, H. B. Wilcox, G. L. Winkelmann, F. Wood.

The following officers were elected for the coming year: Chairman, Fredrick Wood, Hamline University, St. Paul, Minnesota; Secretary, A. L. Underhill, University of Minnesota; an Executive Committee consisting of the Chairman, the Secretary, Gladys Gibbens, University of Minnesota, F. J. Taylor, College of St. Thomas, St. Paul, C. M. Jensen, Macalaster College, St. Paul.

A motion was passed expressing the appreciation of the Section for the hospitality of the College of St. Catherine.

The following seven papers were read:

1. "An integrating operator," by Mr. Max Scherberg, University of Minnesota.

2. "Insect populations," by Mr. John Stanley, University of Minnesota.

3. "Newton's method of solving equations," by Professor W. O. Beal, University of Minnesota.

4. "Developmental geometry," by Miss Marie Ness, Department of Anatomy, University of Minnesota.

5. "Approximate solutions of problems in the calculus of variations," by Professor C. G. Priester, University of Minnesota.

6. "Teaching higher algebra in large classes," by Professor Elizabeth Carlson, University of Minnesota.

7. "A revision of the college entrance requirement in geometry," by Professor Dunham Jackson, University of Minnesota.

Abstracts of these papers follow:

1. The purpose of this paper was to find an operator which would evaluate the indefinite integral of a product $g_1(x)\ g_2(x)$ by operations upon the separate factors of the product. The author found such an operator and gave two examples of its use. In the first example, $g_1(x) = E^{mx}$, $g_2(x) = \sin nx$; in the second, $g_1(x) = E^{\pm x}$, $g_2(x) = \log x$. He also gave a generalized operator for evaluating the indefinite integral of the product, $g_1(x)\ g_2(x) \cdots g_n(x)$.

2. This paper dealt with mathematical considerations of an insect population in which ancestors accumulate as producing members throughout several generations. The author developed a formula for the total number of eggs laid up to and including the kth day of the rth generation and showed how that formula can be used in actual practise to obtain insect population values.

3. Mr. Beal showed by an example how n simultaneous equations involving n variables may be solved to any desired degree of accuracy provided an approximate solution can be obtained by graphical processes or measurements.

4. Miss Ness's paper gave two striking examples of the manner in which certain phenomena of biological development follow simple geometric forms. The first example had to do with cerebral hemispheres and the second with the frontal-fontanelle area.

5. A method of solving certain problems in engineering is based on the consideration of the potential energy of deformation. When a bar is bent by external forces or moments the equation of its elastic curve is that one in which the potential energy of deformation is the least.

The potential energy of deformation may be expressed by the equation

$$U = K \int (d^2y/dx^2)^2 dx$$

and the work due to the external forces or moments by

$$T = P \int f(x)(dy/dx)^2 dx,$$

where P is the external force or moment.

Equating these two expressions and solving for P,

$$P = \frac{K \int (d^2y/dx^2)^2 dx}{\int f(x)(dy/dx)^2 dx}.$$

This equation gives the value of P when the external work and the energy of

deformation are equal. Since the quantity on the right hand side should be a minimum, the problem is to find the equation of the elastic curve which will make it so. An approximate solution may be found by setting up a series of the form

$$y = a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3) + \cdots,$$

in which each term satisfies the boundary condition of the problem and where a_1, a_2, a_3, etc. are parameters to be determined so that the curve produces a minimum value for P. This value of P is a critical value and is on the boundary between stable and unstable equilibrium.

6. In this paper Miss Carlson pointed out some of the differences in methods used in teaching classes of about one hundred students as compared with methods used in teaching small classes. Also, she gave figures showing that the grade of work done by the students in the large classes compared favorably with the grade of work done by the students in the small classes.

7. Mr. Jackson spoke briefly about the organization and purposes of a committee on college entrance requirements in geometry, with regard to which a more detailed announcement appeared in the August-September number of this Monthly.

A. L. UNDERHILL, *Secretary*

THE SIXTH ANNUAL MEETING OF THE INDIANA SECTION

The sixth annual meeting of the Indiana Section of the Mathematical Association of America was held May 3–4, 1929 at Culver Military Academy, Culver, Indiana.

There were sixty present at the meeting including the following twenty-nine members of the Association: W. C. Arnold, Gladys Banes, Stanley Bolks, H. T. Davis, S. C. Davisson, C. S. Doan, J. E. Dotterer, P. D. Edwards, E. D. Grant, H. E. H. Greenleaf, G. E. Happell, C. T. Hazard, F. H. Hodge, H. K. Hughes, Juna M. Lutz, William Marshall, T. E. Mason, H. R. Mathias, G. T. Miller, J. A. Reising, C. K. Robbins, L. S. Shively, J. R. K. Stauffer, R. B. Stone, R. O. Virts, C. J. Waits, K. P. Williams, W. A. Zehring, H. A. Zinszer.

On Friday afternoon at 5:30 a reception was given to the visiting members and their guests. At 6:30 a complimentary banquet which was held in the mess hall was attended by approximately ninety guests of the academy. General L. R. Gignilliat, superintendent of the academy, presided at the banquet and made a brief address of welcome. Entertainment was furnished by Major Norman Imrie, head of the public speaking department of the academy, who regaled the guests with stories appropriate to the occasion. Music was furnished during the banquet by the cadet band.

At eight o'clock a public lecture under the auspices of the academy was given in the gymnasium by Professor Warren Weaver of the University of Wisconsin

on the subject, "Science and Imagination." Professor Weaver presented the new view of mathematical and physical science which is emerging from modern speculations. Ancient mathematics, according to the speaker, made use of defined elements and self-evident axioms. Modern mathematics makes use of undefined elements and assumed postulates. The theorems of modern mathematics are thus creations, not discoveries. Since mathematics is now a product of the creative imagination, it deserves consideration as an art. A somewhat similar change has come about in the logical structure of physical theories. The older model theories of physics explaining by analogy are comparable to the older mathematics; while the modern more abstract physical theories are more closely related with modern postulational mathematics. In the development of physical theories, therefore, the imagination now plays a more significant rôle than formerly and the theories have become more artistic in structure.

At 8:30 Saturday morning, a military review was held in honor of the visitors and this was followed by a tour of the academy buildings.

At the session at 10:00 a.m. in the Memorial Building of the Academy presided over by Professor H. E. H. Greenleaf, De Pauw University, chairman, the following officers were elected: Professor H. A. Zinszer, Hanover College, Chairman; Professor E. D. Grant, Earlham College, Vice-chairman; Professor H. T. Davis, Indiana University, Secretary-treasurer.

A chairman's address was made by Professor Greenleaf on the subject, "Mathematics in the Fundamentals of Music." Professor Greenleaf, considering the musical scale and the principal intervals of music as fundamental, discussed the changes in the scale from earliest times to the present. The speaker pointed out the mathematical basis of the Pythagorean, the diatonic, the mean-tone temperament, and the equal temperament scales and intervals and made a comparison of the four in regard to tonality.

The remainder of the program consisted of the following papers:

1. "The sectioning of freshman engineering students in mathematics," by Professor William Marshall, Purdue University.

2. Extracts from a discourse of J. F. Hennert (Utrecht, 1765): "On the necessity of including the study of mathematics in a good education," by Professor T. E. Mason, Purdue University.

3. "Invariance under the symmetric group of order three of a functional equation due to Abel," by Professor P. D. Edwards, Ball State Teacher's College.

4. "A solution of the biquadratic equation," by Professor E. D. Grant, Earlham College.

5. "A certain general type of contact transformation in three dimensions," by Professor C. K. Robbins, Purdue University.

6. "Three methods for finding the shortest distance between two skew lines" by Margaret L. Darragh, Hanover College (Introduced by Professor Zinszer).

7. "Transformations by reciprocal rays," by Mr. James Avas Cooley, Indiana University (Introduced by Professor H. E. Wolfe).

8. "Some properties of the circles that can be connected with the complete quadrilateral," by Mr. Maurice M. Lemme, De Pauw University (Introduced by Professor Greenleaf).

9. "Present status of the theory of the Volterra integral equation of the second kind," by Professor H. T. Davis, Indiana University.

10. "Notes on quantum mechanics," by Professor H. A. Zinszer, Hanover College.

Abstracts of the papers follow:

1. In the fall of 1928 the mathematics department of Purdue University divided the incoming freshmen in engineering mathematics into three groups: a sub-collegiate group, a normal group, and an advanced or honor group. Professor Marshall set forth in some detail the reasons for this sectioning, how it was done, how the various groups were handled, and the results of the experiment in so far as they are apparent at the present time.

2. Professor Mason's paper consisted of a translation of the inaugural address of J. F. Hennert at Utrecht on a subject of perennial interest to mathematicians. The striking feature of this discourse lies in the fact that the criticisms of students made by this professor in 1765 sound very modern. Apparently students have not changed much in the last century and a half. There are some reasons advanced to the people of the commercial town of Utrecht for the study of mathematics and physics which we should put today under the heading of reasons for the study of engineering.

3. In order that $F(x, y)F(y, z)F(z, x)$ be identical with $F(y, x)F(z, y)F(x, z)$ it is obviously sufficient that $F(x, y)$ be composed of factors which are (1) functions of x only, or (2) functions of y only, or (3) symmetric functions of x and y. That the condition is also necessary is not evident. Proof is given that if F is an algebraic function which is rational, or if irrational, one that belongs to a realm in which the unique factorization law holds, then the conditions named are necessary. Extension is made to the invariance of the function $F(x_1, x_2)$ $F(x_2, x_3) \cdots F(x_n, x_1)$ under the symmetric group of order n.

4. The biquadratic equation is first reduced to the form lacking the term in x^3. The roots are assumed to be $a \pm \sqrt{b}$, $-a \pm \sqrt{c}$. If we express the relation between the roots and the coefficients, there are three equations to solve for a, b, and c. The elimination of b and c leads to an equation of the sixth degree in a, containing the terms a^2, a^4, and a^6. This equation may be solved by Cardan's method in any numerical case; b and c may then be obtained, and the four roots written out.

5. If the transformation $x'=f_1(x, y, z, p, q)$, $y'=f_2(x, y, z, p, q)$, $z'=f_3(x, y, z, p, q)$, $p'=f_4(x, y, z, p, q)$, $q'=f_5(x, y, z, p, q)$ transforms a union of plane elements into a union of plane elements, it is a contact transformation. The analytical condition is that the vanishing of $p'dx'+p'dy'-dz'$ is a consequence of the vanishing of $pdx+qdy-dz$. This leads to a set of four partial differential equations, the integration of which can be determined (theoretically) according to the general theory of such equations. The actual application of this condition

seems to lead to insurmountable difficulties, but certain special cases are of interest. In particular if f_4 and f_5 are functions of p and q only, that is if the orientation of the plane in the transformed element depends only on the orientation of the plane in the original element, the system of partial differential equations can be completely solved. If $f_4 = p$ and $f_5 = q$, and the arbitrary function introduced by integration assumes a certain value, the transformation becomes the well known dilation.

6. The following methods were discussed: I. Through each of the lines any plane is passed perpendicularly. At some position of these planes their line of intersection will intersect each of the skew lines. The distance between these points is the shortest distance between the two lines. II. A plane is constructed perpendicular to one of the lines and through each line a plane is passed perpendicular to it. By expressing the equations for these two planes in their normal form and adding the right members the distance is obtained. III. The last method finds the distance directly by the minimizing process.

7. Laguerre in Nouvelles Annales, 3rd series, volume 1, defined the transformation by reciprocal rays and gave some of its properties. Mr. Cooley introduced a new constant for the modulus of the transformation, defining it as a cross ratio. In terms of the constant, he developed relations between the angle which reciprocal rays make with each other and the axis of transformation. He also gave additional properties and applications of the transformation especially with regard to circles and their tangents.

8. This paper proved by methods of Euclidean geometry alone the following theorem given by Jakob Steiner in the Annales de Gergonne, vol. 18, p. 16: In each of the four triangles formed by the sides of a complete quadrilateral there is one circle inscribed and three circles escribed, making in all sixteen circles. The centers of these sixteen circles arrange themselves in groups such that each of the four circles of one group cuts orthogonally all the circles of the other group. The lines of centers of the two groups of circles are perpendicular to each other. The lines of centers of the two groups of circles intersect at the point of intersection of the circles circumscribed to the four triangles forming the quadrilateral. Various consequences of the theorem were also exhibited by the speaker.

9. The methods used to solve the Volterra integral equation, not only for the case of continuous kernels, but for various types of discontinuities, were discussed. Numerous properties of the equation of the closed cycle, namely the case of the kernel of the form $K(x-t)$, were exhibited.

10. Assuming a closed system consisting of a nucleus and an electron the former being a point charge located at the origin of coordinates, the principle of the conservation of energy expressed in Newtonian notation was imposed. Applying the principle of Maupertuis (least action) and assuming the resulting equation to describe a family of wave-fronts travelling with a speed $E/\sqrt{[2m(E-V)]}$, where E is the total energy and V the potential energy function, a particular form of de Broglie's wave equation finally resulted.

At the afternoon session a resolution was adopted by the members of the section expressing their appreciation of the welcome that had been given them by the academy and of the efforts of General Gignilliat and of Major G. H. Crandall, Captain L. R. Kellam and other members of the department of mathematics who had contributed to the success of the meeting.

H. T. DAVIS, *Secretary*

A MODIFICATION OF A PROOF BY STEINER

By OTTO DUNKEL, Washington University

INTRODUCTION. An elegant and elementary proof was given by Steiner of the theorem that the equilateral triangle has the greatest area of all triangles having the same perimeter.[1] This proof is interesting in that no use is made of either parallels or metrical expressions for the area; it applies therefore whether the sum of the angles of a triangle is supposed to be less than, equal to, or more than 180°, and Steiner showed that his proof applied to spherical triangles without essential change. His proof consists of two parts of which the first part is essentially the proof under Theorem I below, while the second part has been altered to the form of proof under Theorem II. This modified form of proof is applicable to other similar geometrical theorems, and two such theorems are proved in this way without the use of parallels or metrical expressions for the area. The following proofs are worded for spherical triangles since in a few places restrictions are required peculiar to this form of geometry. For the cases where the sum of the angles of the triangle is less than or equal to 180° the proofs are essentially the same but simpler. In conclusion two theorems are given which result from the consideration of a metrical expression for the area. In the discussions below when one side of a triangle is designated as a base the term side will be considered to apply only to the two remaining sides.

THEOREM I. *Two triangles which have equal perimeters and bases of equal lengths have unequal areas if they are neither congruent nor symmetric. The triangle having the smaller area has the smallest base angle, the greatest base angle, the shortest side and the longest side.*

PROOF: Let ABC and $A'B'C'$ be two triangles which are neither congruent nor symmetric, but are such that $AB=A'B'$, $AC+BC=A'C'+B'C'$, $A \leqq B$, $A' \leqq B'$, where A denotes the angle BAC etc. The equality signs in the last relation are assumed to hold for only one triangle, for otherwise the two triangles would be congruent. Let the bases be made to coincide so that A' falls at A and B' at B. If then C and C' fall on opposite sides of the common base, we shall replace one triangle by its symmetric triangle and we shall suppose that the lettering of the vertices of the new triangle is the same as that for the old.

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[1] Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphère et dans l'espace en général*, Crelle's Journal, vol. 24 (1842), pp. 96–99.

The new triangle has the same area and parts as the old. With this understanding C and C' lie on the same side of the base, and C' cannot fall within ABC or upon a side, for then we should have $AC'+BC'<AC+BC$; for the same reason C cannot fall within ABC' or upon a side. It follows then that a longer side of one triangle, say AC', must cut in a point M a shorter side, BC, of the other triangle, and the point M must lie within each of the segments AC' and BC. Hence $\angle BAC'<\angle BAC\leqq\angle ABC<\angle ABC'$, and therefore $BM<AM$. On MA lay off $MD=MB$, and on MC lay off $ME=MC'$, and draw DE. The two triangles MBC' and MDE have equal areas and $DE=BC'$. It will be shown that E lies within the segment MC. We have $AC+CM+MB=AD+DM+MC'+C'B$, and, since $MB=DM$, $MC'=ME$, $C'B=DE$, this equality reduces to $AC=AD+DE+(ME-CM)$. If $ME\geqq CM$ we would have the length of the broken line $ADEC$ equal to the length of the unbroken line AC, which is impossible. Hence $CM>ME$. It now follows that the area of ABC exceeds the area of ABC' by the area of the quadrilateral $ADEC$. Moreover, $BC'<BM+MC'=BM+ME<BC$, and from this inequality follows that $AC'>AC$. The theorem will be used in the following form:

If two triangles have bases equal to c and sides $a<b$, $a'\leqq b'$, respectively, such that $a+b=a'+b'$; then, if $a<a'$, the triangle with the side, a, has a smaller area than the triangle with the side a'.

THEOREM II: *Of all triangles having the same length of perimeter the equilateral triangle has the greatest area.*

PROOF: Let a, b, c be the lengths of the sides of a triangle which is not equilateral, and let S be its area; let e be the side of an equilateral triangle having the area E, and such that $3e=a+b+c$. Let a and c be the shortest and longest side, respectively; then $a<e<c$. Consider an isosceles triangle with the base c and the equal sides $a'=\frac{1}{2}(a+b)$, and let its area be S'. If $a=b=a'$, then $S'=S$; but if a and b are unequal, then $a<a'<b\leqq c$. Hence $S'>S$. Since $2a'+c=3e$ and $c>e$, we have $a'<e<c$. Consider now a triangle with base a', one side of length e and the other side of length b' so that $a'+b'=2e$, and let its area be S''. Since $a'<e$, it follows that $e<b'$. Hence in the two triangles S' and S'' having bases of length a' the shorter side of the first, a', is less than the shorter side, e, of the second, and therefore $S''>S'$. We compare finally E and S'' considered as having the base e. From the inequalities above $a'<b'$, $a'<e$. Hence $E>S''$, and $E>S''>S'\geqq S$, and the theorem is proved.

THEOREM III: *If two triangles which circumscribe the same small circle have an angle of one equal to an angle of the other, the one having the shortest adjacent side has the longest adjacent side and the greater area.*

PROOF: Let ABC and $A'B'C'$ be two triangles circumscribing the same small circle with center I, and such that $C=C'$, $CB<CA$, $C'B'\leqq C'A'$, $CB<C'B'$. Let the triangles be placed so that C and C' coincide, and if in this position $C'B'$ does not fall along CB we shall replace one triangle by its symmetric triangle so that this will be the case. Let L and M be the points of contact of CB and CA,

and N and N' the points of contact of AB and $A'B'$. Let $\angle LIM=\gamma$, $\angle MIN=\alpha$, $\angle NIL=\beta$, $\angle MIN'=\alpha'$, $\angle N'IL=\beta'$. Then $\alpha+\beta=\alpha'+\beta'$. In the right triangles LIB and LIB', the common side IL is less than 90° and $180°>LB'>LB$; hence $\frac{1}{2}\beta<\frac{1}{2}\beta'$ and therefore $\alpha'<\alpha$. In the right triangles MIA and MIA' we have again the common side $MI<90°$, $\angle MIA>\angle MIA'$; therefore $MA>MA'$ or $CA>C'A'$. Hence AB and $A'B'$ meet in a point P which lies within each segment, and it is easily shown that $PN=PN'$. In the triangles BIN and BIP with the common side BI and the angle at B in common, $\angle BIN=\frac{1}{2}\beta<\frac{1}{2}\beta'=\angle BIP$, and hence P lies within the segment NA. We show in a similar manner that P lies within the segment $N'B'$. We have then $AN=AM>A'M=A'N'\geqq B'L=B'N'$. Hence $AP=AN-NP>B'N'-PN'=B'P$. Also $A'N'=A'M\geqq B'L>BL=BN$, and hence $A'P=A'N'+N'P>BN+NP=BP$. Thus in the two triangles $PA'A$ and PBB' with equal angles at P, $A'P>BP$ and $AP>B'P$. Hence the area of the first triangle is greater than the area of the second, and from this follows at once that the area of ABC is greater than the area of $A'B'C$ or of $A'B'C'$. The angles α, β, γ for the triangle ABC will be designated as its central angles.

COROLLARY: *Of two triangles circumscribing the same circle and having a central angle of one equal to a central angle of the other, the one with the smallest of the remaining four central angles has the greatest central angle and the greater area.*

This follows at once from the above since $\beta<\beta'\leqq\alpha'<\alpha$ and $\gamma=\gamma'$.

THEOREM IV: *If two triangles circumscribe the same small circle and have an angle of one equal to an angle of the other, the one having the smallest adjacent side has the longest adjacent side and the greater perimeter.*

PROOF: The assumptions here and the proof of the first part are the same as in the proof of III, and it remains to prove that the perimeter of ABC is greater than the perimeter of $A'B'C'$. We have $\angle A'IA=\frac{1}{2}(\alpha-\alpha')$, $\angle B'IB=\frac{1}{2}(\beta'-\beta)$, and therefore $\angle A'IA=\angle B'IB$, since $\alpha+\beta=\alpha'+\beta'$. In the triangle AIB, $IN<90°$ and $AB<180°$. Hence AI produced meets ANB produced in A_1 so that $AIA_1=ANA_1=180°$, and since $NB<NA_1<180°$, we have $IB<IA_1$, or $AI+IB<AI+IA_1=180°$. Lay off on MA the lengths $MB_1=LB$ and $MB_1'=LB'$, and draw IB_1 and IB_1'. Then in the triangle AB_1I, $B_1I+IA<180°$, and $\angle AIA'=\angle B_1'IB_1\leqq\frac{1}{2}\angle AIB_1<90°$. Therefore $AA'>B_1'B_1=BB'$* In the two triangles ABC and $A'B'C$ the part $CM+CL$ is common and the corresponding remaining parts of the perimeters are $2(MA+LB)$ and $2(MA'+LB')$. But $MA+LB=MA'+A'A+LB'-BB'>MA'+LB'$. Hence the perimeter of ABC is greater than the perimeter of $A'B'C'$.

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* This theorem has been proposed as problem 3331 [1928, 321] in this Monthly, and a proof will appear later. The theorem is true whether the sum of the angles of the triangle is more than, equal to, or less than 180°; but in the last two cases the condition on the sides is of course omitted and the proof is much simpler.

COROLLARY: *Of two triangles circumscribing the small circle and having a central angle of one equal to a central angle of the other, the one with the smallest of the four remaining central angles has the greatest of the four and the greater perimeter.*

THEOREM V. *Of all triangles circumscribing the same small circle, the equilateral triangle has the smallest area and the shortest perimeter.*

PROOF: Let ABC be a triangle which is not equilateral and which circumscribes a given small circle, and let its area be S and its perimeter, p. Then the central angles α, β, γ are not all equal and we may suppose that the smallest is α and that the largest is γ. Hence $\alpha<120°<\gamma$. Consider the circumscribing triangle with a central angle α and $\beta'=\gamma'=180°-\frac{1}{2}\alpha$. Then $\beta'=\gamma'>120°>\alpha$. If $\beta=\gamma$, the two triangles are congruent and the area S' and the perimeter p' of the second are the same as for the first. If $\beta\neq\gamma$, $\beta<\gamma$ and since $\beta+\gamma=2\beta'=2\gamma'$, we have $\beta<\beta'$ and $\gamma>\gamma'=\beta'>\beta$. Hence $S>S'$ and $p>p'$. Now consider the triangle with central angle $\beta''=\beta'>120°$, $\gamma''=120°$, $\alpha''=240°-\beta''<120°$, and let S'' and p'' denote its area and perimeter. The two triangles S' and S'' have the central angle β' and of the remaining four angles $\alpha<\gamma'$, $\alpha''<\gamma''$, and $\alpha<\alpha''$. The last inequality follows form $\alpha''+\gamma''=\alpha+\gamma'$, $\gamma''=120°<\gamma'$. Hence $S'>S''$ and $p'>p''$. Now the circumscribing equilateral triangle has $\alpha'''=\beta'''=\gamma'''=120°$. Let its area be E and its perimeter $3e$. Then S'' and E have $\gamma''=\gamma'''=120°$, and of the four remaining angles $\alpha''<120°<\beta''$; hence $S''>E$ and $p''>3e$ and $S\geqq S'>S''>E$ and $p\geqq p'>p''>3e$, and the theorem follows.

Some special forms of proof: In spherical geometry the properties of polar triangles and the measure of the area of a triangle enable us to prove the following theorem:

THEOREM VI: *Of all triangles inscribed in a given small circle on the surface of a sphere the equilateral triangle has the greatest area and the greatest perimeter.*

PROOF: Let S be the area of a triangle with the angles A, B, C and the opposite sides a, b, c, inscribed in a circle with the center I and of radius $R<90°$. Let E be the area of the inscribed equilateral triangle with the equal angles ϕ and the equal sides e. The polar triangle of S has the angles $180°-a$, $180°-b$, $180°-c$; the sides $180°-A$, $180°-B$, $180°-C$; and the area S'. The polar of E has the equal angles $180°-e$, the equal sides $180°-\phi$, the area E'. The polar triangle of S circumscribes the circle with center I and with the radius $90°-R$ and similarly for the polar of E. Hence by V we have $S'>E'$ or $540°-(a+b+c)>540°-3e$. Therefore $3e>a+b+c$. Also by the same theorem $540°-(A+B+C)>540°-3\phi$ or $3\phi>A+B+C$. Hence $E>S$.

In Euclidean geometry of the plane the area of a triangle is equal to the product of the radius of the inscribed circle and one half of the perimeter. Hence in this geometry we can infer[1] II from V or V from II.

[1] This is shown in the article, *Maximum and minimum areas in geometry*, School Science and Mathematics, vol. 28 (1928), pp. 710–716.

A GENERALIZATION OF THE ORTHOPOLE THEOREM

By R. GOORMAGHTIGH, Bruges, Belgium

1. If A_1, B_1, C_1 be the projections of the vertices A, B, C of the triangle ABC on a straight line Δ, the perpendiculars from A_1 on BC, from B_1 on CA, and from C_1 on AB are concurrent at a point called the orthopole[1] of Δ for the triangle ABC.

More generally, when from A, B, C straight lines are drawn making with Δ equal angles θ at α, β, γ, the straight lines drawn from α, β, γ and making angles equal to $\pi-\theta$ with BC, CA, AB, respectively, are also concurrent.[2]

The orthopole theorem has further also been generalized as follows:

Let A_1', B_1', C_1' be the points dividing AA_1, BB_1, CC_1 in the same ratio; then the perpendiculars from A_1', B_1', C_1' on BC, CA, AB respectively, are concurrent.

We shall now deal with a new theorem containing these properties and also other known theorems in the geometry of the triangle.

2. Take Δ as x-axis, the y-axis being any perpendicular on Δ, and let (x_1,y_1) (x_2,y_2), (x_3,y_3), be the coordinates of A, B, C; call

$$X_1 = x_2 - x_3, \quad X_2 = x_3 - x_1, \quad X_3 = x_1 - x_2,$$
$$Y_1 = y_2 - y_3, \quad Y_2 = y_3 - x_1, \quad Y_3 = y_1 - y_2,$$

Let now $A\alpha, B\beta, C\gamma$ be the straight lines drawn through A, B, C, and making with Δ the angle θ at α, β, γ; if $\tan\theta = l$, the coordinates of α are $(x_1 - l^{-1}y_1, 0)$ and those of the point α' dividing $A\alpha$ so that $\alpha'A : \alpha A = \mu$ are $x_1 - l^{-1}\mu y_1$, $(1-\mu)y_1$.

The equation to the straight line d_a drawn through α' and making the angle ϕ with BC is:

$$[y - (1-\mu)y_1] = m(x - x_1 + l^{-1}\mu y_1), \tag{1}$$

and, if $\tan\phi$ is denoted by k,

$$m - (Y_1/X_1) = k[1 + (mY_1/X_1)].$$

Hence (1) becomes

$$p_1x + q_1y + r_1 \equiv (kX_1 + Y_1)x + (kY_1 - X_1)y + (1 - \mu + l^{-1}\mu k)y_1X_1 + (l^{-1}\mu - k + k\mu)y_1Y_1 - kxX_1 - x_1Y_1 = 0. \tag{2}$$

If β' and γ' are the points dividing $B\beta$ and $C\gamma$ so that

$$\beta'B : \beta B = \gamma'C : \gamma C = \mu,$$

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[1] Neuberg, *Nouvelle Correspondance Mathématique* (1875), p. 189; Desboves, *Questions de Géométrie* (1875), p. 241; Gallatly, *The Modern Geometry of the Triangle*, 2nd ed., p. 46. On the properties of the orthopole, see my own paper in the Tôhoku Mathematical Journal, vol. 30 (1926), pp. 77–125.

[2] J. A. Third, Proceedings of the Edinburgh Mathematical Society (1913), p. 17; Neuberg, Mathesis (1914), p. 89.

the equations to the straight lines d_b, d_c drawn through β' and γ', and making with CA and AB the angle ϕ are similar to (2).

The area σ of the triangle formed by d_a, d_b, d_c is given by the formula

$$2\sigma = \frac{|\,p_1\ q_1\ r_1\,|^2}{(p_2q_3 - p_3q_2)(p_3q_1 - p_1q_3)(p_1q_2 - p_2q_1)}.$$

But

$$p_2q_3 - p_3q_2 = p_3q_1 - p_1q_3 = p_1q_2 - p_2q_1 = (k^2+1)\sum x_1Y_1 = 2(k^2+1)S,$$

S being the area of ABC.

Further if to the last row of the determinant $|p_1\ q_1\ r_1|$ is substituted the sum of the three rows, the determinant becomes

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ 0 & 0 & \Phi \end{vmatrix} = (p_1q_2 - p_2q_1)\Phi$$

where

$$\Phi = (1 - \mu + l^{-1}\mu k)\sum y_1X_1 - \sum x_1Y_1 = 2(2 - \mu + l^{-1}\mu k)S.$$

Therefore

$$\sqrt{\frac{\sigma}{S}} = (2 - \mu + \mu \tan\phi \cot\theta)\cos\phi.$$

Hence the following theorem:

Let $A\alpha$, $B\beta$, $C\gamma$ be the straight lines drawn through the vertices A, B, C, of the triangle ABC and making with a given straight line Δ, the same angle θ at α, β, γ. If α', β', γ' are the points dividing $A\alpha$, $B\beta$, $C\gamma$ in the same ratio, so that

$$\alpha'A:\alpha A = \beta'B:\beta B = \gamma'C:\gamma C = \mu,$$

the lines drawn through α', β', γ' and making with BC, CA, AB the same angle ϕ, form a triangle similar to ABC, and, for given angles θ and ϕ and a given ratio μ, the ratio of similitude of the two triangles is the same for any position of Δ in the plane of ABC.

The value of the ratio of similitude is

$$(2 - \mu + \mu \tan\phi \cot\theta)\cos\phi.$$

3. The orthopole theorem corresponds to $\mu = 1, \theta = \phi = \pi/2$; similarly the second theorem in paragraph 1 corresponds to $\mu = 1, \phi = \pi - \theta$. If $\phi = \theta$, the ratio of similitude is $2\cos\phi$, for any value of μ. When $\phi = 0$, the ratio of similitude is $2 - \mu$; hence the following theorem:

Let $A\alpha$, $B\beta$, $C\gamma$ be three parallels drawn through the vertices of ABC and cutting a given straight line at α, β, γ. If α', β', γ' be the points dividing $A\alpha$, $B\beta$, $C\gamma$, in the same ratio so that

$$\alpha' A : \alpha A = \beta' B : \beta B = \gamma' C : \gamma C = \mu,$$

then the ratio of similitude of the triangle formed by the parallels to BC, CA, AB drawn through α', β', γ' and the triangle ABC is $2-\mu$.

When $\alpha' \equiv \alpha$, $\beta' \equiv \beta$, $\gamma' \equiv \gamma$, these two triangles are equal.[1]

THEOREMS ON TOTAL REPRESENTATIONS AS SUMS OF SQUARE OR TRIANGULAR NUMBERS

By E. T. BELL, California Institute of Technology

1. We give further instances of a curious type of theorem considered in a previous note.[2] To recall what was proved, let $E(n)$, $O(n)$ denote respectively the total numbers of representations of the positive integer n as a sum of an even, an odd number of squares with roots $\lesseqgtr 0$, the order of the squares in each representation being essential. It was shown that $E(2n) > O(2n)$, $O(2n+1) > E(2n+1)$.

The following definitions will be required. The nth triangular number, where n is an integer >0, is $n(n+1)/2$. If for every pair of co-prime integers a, $b>0$, the function $g(n)$, single-valued and finite for all integers $n>0$, has the property $g(ab)=g(a)g(b)$, $g(n)$ is said to be factorable. Neither of $O(n)$, $E(n)$ is factorable.

2. Let $Q_r(n)$ denote the number of representations of the integer $n>0$ as a sum of r squares with roots $\lesseqgtr 0$, and $T_r(n)$ the number of representations of n as a sum of r triangular numbers. Write

$$Q(n) \equiv Q_1(n) - \tfrac{1}{2}Q_2(n) + \tfrac{1}{3}Q_3(n) - \cdots + (-1)^{n-1} n^{-1} Q_n(n),$$

$$T(n) \equiv T_1(n) - \tfrac{1}{2}T_2(n) + \tfrac{1}{3}T_3(n) - \cdots + (-1)^{n-1} n^{-1} T_n(n).$$

Neither $Q_j(n)$ nor $T_j(n)$ for n arbitrary and j odd, >1, or j even, >8, is expressible as a polynomial in the real divisors of n alone. Further, the algebraical complexity of the functions expressing $Q_j(n)$, $T_j(n)$ increases rapidly with j. The following is thus unexpectedly simple.

Theorem I. *If $2^c (c \geqq 0)$ is the highest power of 2 that divides n, each of the functions*

$$\frac{(-1)^{n-1}}{2} Q(n), \quad \frac{2^c T(n)}{3-2^{c+1}}$$

is equal to the sum of the reciprocals of all the odd divisors of n, and hence each is factorable.

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[1] Neuberg, Wiskundig Tydschrift (1913–1914), p. 79, Thébault, Journal de Mathématiques Élémentaires (1913–1914), p. 121. The analogon for the tetrahedron had already been given by Neuberg in Mathesis (1891), p. 50.

[2] This Monthly, vol. 30 (1923), p. 441.

Assuming this for a moment we have at once the following analogues of the theorems on $O(n)$, $E(n)$. Write

$$Q'(n) \equiv Q_1(n) + \tfrac{1}{3}Q_3(n) + \tfrac{1}{5}Q_5(n) + \cdots,$$

$$Q''(n) \equiv \tfrac{1}{2}Q_2(n) + \tfrac{1}{4}Q_4(n) + \tfrac{1}{6}Q_6(n) + \cdots,$$

$$T'(n) \equiv T_1(n) + \tfrac{1}{3}T_3(n) + \tfrac{1}{5}T_5(n) + \cdots,$$

$$T''(n) \equiv \tfrac{1}{2}T_2(n) + \tfrac{1}{4}T_4(n) + \tfrac{1}{6}T_6(n) + \cdots.$$

Theorem II. *$Q'(n)$ is greater or less than $Q''(n)$ according as n is odd or even, and similarly for $T'(n)$, $T''(n)$.*

The next states several further consequences of theorem I, all of which can be verified at a glance from the assumed truth of theorem I.

Theorem III. *If a, b are coprime,*

$$Q(ab) = \tfrac{1}{2}(-1)^{(a-1)(b-1)}Q(a)Q(b), \quad T(ab) = T(a)T(b).$$

If $n, \alpha, \beta, \gamma, \cdots, \delta$ are arbitrary integers >0, and $p, q, \cdots, r$ are g distinct odd primes, and if c is an arbitrary integer $\geqq 0$,

$$Q(2n) = (-1)^n Q(n), \quad Q(2^\alpha) = -2, \quad Q(1) = 2,$$

$$Q(2^c p^\beta q^\gamma \cdots r^\delta) = \frac{(-1)^{n-1}}{2^{g-1}} Q(p^\beta)Q(q^\gamma)\cdots Q(r^\delta),$$

$$T(2^c) = \frac{3-2^{c+1}}{2^c},$$

$$T(2^c p^\beta q^\gamma \cdots r^\delta) = T(2^c)T(p^\beta)T(q^\gamma)\cdots T(r^\delta).$$

To prove theorem I, and hence all, it will be sufficient to prove the part referring to $Q(n)$, as that for $T(n)$ proceeds in exactly the same way from the appropriate identity, which will be indicated. It is more interesting, however, to outline the proof of a more general theorem, from which the part relating to $Q(n)$ follows immediately.

3. Let $f(x)$ be single-valued and finite for integer values of x, and such that $f(n)=f(-n)$ for all integers n. Write

$$F_j(n) \equiv \sum f(n_1 + n_2 + \cdots + n_j),$$

where the summation refers to all integers $n_i \gtrless 0 (i=1, \cdots, j)$ such that

$$n_1^2 + n_2^2 + \cdots + n_j^2 = n,$$

and write

$$F(n) \equiv F_1(n) - \tfrac{1}{2}F_2(n) + \tfrac{1}{3}F_3(n) - \cdots + \frac{(-1)^{n-1}}{n}F_n(n).$$

Denote by $\sigma(n)$ the sum of all the divisors of n.

Theorem IV. *The summation on the right referring to all pairs* (t, τ) *of integers* $t>0, \tau>0, \tau$ *odd, such that* $n=t\tau$,

$$F(n) = -1/n\left[\left\{1+(-1)^n\right\}\sigma(\tfrac{1}{2}n)f(0) + 2\sum(-1)^t\tau f(t)\right].$$

To see first that this implies the $Q(n)$ part of theorem I, take $f(x)=1$ for all integer values of x, as clearly is permissible under the definition of $f(x)$. A short reduction of the resulting right hand member gives the required relation.

To prove theorem IV, observe that we may take $f(n)\equiv\cos nx$, where x is a parameter, for all integers n, and get a true theorem provided theorem IV is true. But conversely, if the cosine form of the theorem is an identity in x, we can infer the general form as stated.[1] The cosine form, however, follows by a straightforward reduction from the identity.[2]

$$\log(1+\sum{}'q^{n^2}) = \log\theta_3 + 2\sum\frac{(-1)^nq^n}{n(1-q^{2n})}\quad(1-\cos 2nx),$$

where Σ' refers to $n=\pm1, \pm2, \pm3, \cdots$, Σ to $n=1, 2, 3, \cdots$, and

$$\log\theta_3 = \sum\left[\log(1-q^{2n}) + 2\log(1+q^{2n-1})\right],$$

where Σ refers to $n=1, 2, 3, \cdots$. The expansions are valid for q, x suitably restricted, and similarly for the series obtained by expanding the logarithms by the logarithmic series. Comparison of coefficients of like powers of q in the result gives the stated cosine identity, as can be easily verified.

There is a similar but more complicated generalization of the $T(n)$ part of theorem I. Omitting this, we need only state the classic identity which implies the theorem as stated:

$$1+\sum_{n=1}^{\infty}q^{n(n+1)/2} = \Pi(1-q^n)\Pi(1+q^n)^2,$$

from which, by taking logarithms and expanding, the result follows.

ON CERTAIN FUNCTIONAL RELATIONS

By MORGAN WARD, California Institute of Technology

1. *Introductory problem.* If $y=f(x)$ is an analytic function of x for $0\leqq|x|<r$ and if $f(0)=0, f'(0)\neq0$ so that

$$y = a_1x + a_2x^2 + a_3x^3 + \cdots \qquad (a_1\neq0), \tag{1}$$

then the inverse function $x=f^{-1}(y)$ is also analytic for $0\leqq|y|<\rho=\rho(r)$ and vanishes with y.

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[1] This is a simple instance of what was called paraphrase in several previous papers, e.g., Transactions of the American Mathematical Society, vol. 22 (1921), p. 1.

[2] See almost any text on elliptic functions, e.g., Tannery-Molk, vol. 3, p. 116.

Suppose that the function $f^{-1}(x)$ is identical with[1] $f(x)$, so that

$$x = a_1y + a_2y^2 + a_3y^3 + \cdots . \tag{2}$$

The coefficients $a_1, a_2, a_3, \cdots$, must then satisfy certain algebraic conditions. These conditions express the fact that the result of substituting for the successive powers of x in (1) their expressions in terms of y from (2) must reduce to the identity $y=y$. In particular, we see that $a_1^2=1$. There are two totally different cases according as $a_1=+1$ or $a_1=-1$. If $a_1=+1$, the remaining coefficients $a_2, a_3, \cdots$, all vanish; if however $a_1=-1$, the situation is more complicated. We find that

$$a_3 = -a_2^2, \quad a_5 = 2a_2^4 - 3a_2a_4$$
$$a_7 = -13a_2^6 + 18a_2^3a_4 - 4a_2a_6 - 2a_4^2$$
$$a_9 = 145a_2^8 - 221a_2^5a_4 + 35a_2^3a_6 + 50a_2^2a_4^2 - 5a_2a_8 - 5a_4a_6$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x^n = (-y)^n\left\{1 - na_2y + \frac{n(n+1)}{2!}a_2^2y^2 - \left(\frac{n(n-1)(n+4)}{3!}a_2^3 + na_4\right)y^3 + \left(\frac{n(n-3)(n+2)(n+7)}{4!}a_2^4 + n(n+2)a_2a_4\right)y^4 - \cdots\right\}$$

By a somewhat lengthy induction, we can establish the following theorem:

Theorem 1: *Given*

$$y = f(x) = a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$x = f(y) = a_1y + a_2y^2 + a_3y^3 + \cdots \qquad (a_1^2 = 1)$$

Then if $a_1=1$,

$$a_n = 0 \quad (n = 2,3,4, \cdots,)$$

But if $a_1=-1$,

$$a_{2n+1} = P_n(a_2, a_4, \cdots, a_{2n}) = P_n(-a_2, -a_4, \cdots, -a_{2n}),$$

where P_n *is a uniquely determined polynomial in* $a_2, a_4, \cdots, a_{2n}$ *with integral coefficients.*[2]

The following result is immediate.

Theorem 2: *The necessary and sufficient condition that* x *and* y *be related as in theorem* 1 *is that there exist an analytic function* $F(x, y)$ *of* x *and* y *satisfying the conditions,*

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[1] A simple example of such a function is $y=x/(x-1)=-x-x^2-x^3-\cdots$.

[2] I have not yet succeeded in obtaining the explicit expression for P_n.

$$F(x,y) = F(y,x,) = F(0,0) = 0\,; \quad F_x(0,0) \neq 0.$$

2. *Extended problem.* Suppose that $y=\Phi(a, b; x)$ is a function of x and the two real parameters a and b which satisfies the following conditions:
(i) $\Phi(a, b; x)$ *is an analytic function of* x *on and within the square* $\mathfrak{S}$ *bounded by the lines* $a=\pm h$, $b=\pm h$ *in the* $a\cdot b$ *plane for* $0 \leqq |x| < r = r(h)$;
(ii) $\Phi(a, b; x)$ *vanishes with* x *throughout* $\mathfrak{S}$.

We may consequently write

$$y = \Phi(a,b\,;\,x) = \phi_1(a,b)x + \frac{\phi_2(a,b)x^2}{2!} + \frac{\phi_3(a,b)}{3!}x^3 + \cdots,$$

where

$$\phi_n(a,b) = (\partial^n\Phi/\partial x^n).$$

If, moreover, $\phi_1(a, b)\neq 0$ in $\mathfrak{S}$, the inverse function $x=\Phi^{-1}(y)$ exists for $0 \leqq |y| < \rho(h)$. Let us assume finally that
(iii) $\Phi^{-1}(x)=\Phi(b, a; x)$ for $0 \leqq |x| < r(h)$ throughout $\mathfrak{S}$. We shall then have

$$x = \Phi(b,a\,;y) = \phi_1(b,a)y + \frac{\phi_2(b,a)y^2}{2!} + \frac{\phi_3(b,a)y^3}{3!} + \cdots$$

and as in section 1 it is necessary that

$$\phi_1(a,b)\cdot\phi_1(b,a) = 1. \tag{3}$$

Let us determine some of the properties of functions which satisfy the conditions (i), (ii), and (iii). We shall refer to such functions as "Φ-functions."

3. *Canonical form for functions.* From (3) we see that $\phi_1(a, b)$ and $\phi_1(b, a)$ can never vanish in $\mathfrak{S}$ and must both be of the same sign in $\mathfrak{S}$. If we write

$$y = |\,\phi_1(a,b)\,|^{1/2}v, \quad x = |\,\phi_1(b,a)\,|^{1/2}u,$$

the series defining Φ in section 2 become either

$$\begin{aligned} v &= u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned} \tag{I}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are positive in $\mathfrak{S}$, or

$$\begin{aligned} v &= -u + \psi_2u^2 + \psi_3u^3 + \cdots \\ u &= -v + \psi_2'v^2 + \psi_3'v^3 + \cdots \end{aligned} \tag{II}$$

if $\phi_1(a, b)$ and $\phi_1(b, a)$ are negative in $\mathfrak{S}$; where in both cases

$$\begin{aligned} \psi_n = \psi_n(a,b) &= \frac{\phi_n(a,b)}{n!}\,|\,\phi_1(b,a)\,|^{(n+1)/2} \\ \psi_n' = \psi_n(b,a) &= \frac{\phi_n(b,a)}{n!}\,|\,\phi_1(a,b)\,|^{(n+1)/2} \end{aligned} \tag{4}$$

$$(n = 2,3,\cdots,).$$

If we write $a=b$ in (I) and (4), we see from the first part of theorem 1 that

$$\phi_n(a,a) = 0. \qquad (n = 2,3, \cdots,).$$

From (3) and (4) follows

Theorem 3. *If $y=F(a, b; x)$ is any Φ-function whose coefficients are polynomials in a and b, then the coefficient of every power of x in F save the first is divisible by $a-b$.*

The two canonical forms of Φ-function in (I) and (II) show us that we have a correspondence with the two types of solution of $y=f(x)$, $x=f^{-1}(y)$ in theorem 1. Let us call Φ-functions of the first type "proper functions" and Φ-functions of the second type "improper functions." $y=x$ is the simplest proper function, but in contrast to the first part of theorem 1, we have a theorem analagous to theorem 2 for both proper and improper functions. We shall confine our statement to the former type of function in the canonical form (I).

Theorem 4. *The necessary and sufficient condition that v be a proper Φ-function of u is that u and v be connected by an implicit relation of the form*

$$u - v + F(a,b\,;u,v) - F(b,a\,;v,u) = 0, \tag{5}$$

where for sufficiently small positive values of $|u|$ and $|v|$, $F(a, b; u, v)$ is an analytic function of both u and v in some region $\mathfrak{R}$ in the ab-plane which includes the origin, and where

$$F(a,b\,;0,0) = F(b,a\,;0,0),$$
$$F_x(a,b\,;0,0) = F_x(b,a\,;0,0),$$
$$F_y(a,b\,;0,0) = F_y(b,a\,;0,0),$$

for all values of a and b in $\mathfrak{R}$.

In fact these conditions allow us to substitute for v a series in u with undetermined coefficients Ψ_n which we know will be convergent, and to determine the $\Psi_n=\Psi_n(a, b)$ by equating the coefficients of $u, u^2, u^3, \cdots$, to zero in the resulting identity; in particular, we shall have $\Psi_1=1$. Now if instead we substitute for u a series in v with undetermined coefficients Ψ_n', the equations determining Ψ_n' are obtained from those determining Ψ_n by merely inter-changing a and b, so that $\Psi_n'(a, b)=\Psi_n(b, a)$ and v is a proper function of u. Conversely, if (I) holds, by halving and subtracting the two series we obtain

$$u - v + \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(a,b)u^n - \tfrac{1}{2}\sum_{n=2}^{\infty}\psi_n(b,a)v^n = 0,$$

where, by our definition of a Φ-function, $\frac{1}{2}\sum_{n=2}^{\infty}\psi_n(a, b)\,u^n$ satisfies all the conditions imposed upon $F(a, b, u, v)$ in the theorem.

5. *Example.* As an illustration of a proper Φ-function, suppose temporarily that a, b are real, but never zero. Consider the relation

$$(1 + bx)^a = (1 + ay)^b. \tag{6}$$

By the binomial theorem, if $|x|<1/|b|$, $|y|<1/|a|$

$$x - y + \frac{(a-1)bx^2}{1\cdot 2} + \frac{(a-1)(a-2)b^2x^3}{1\cdot 2\cdot 3} + \cdots$$
$$- \frac{(b-1)\cdot ay^2}{1\cdot 2} - \frac{(b-1)(b-2)a^2y^3}{1\cdot 2\cdot 3} - \cdots = 0,$$

so that

$$x - y + F(a,b\,;x,y) - F(b,a\,;y,x) = 0,$$

where

$$F(a,b\,;x,y) = \sum_{n=1}^{\infty} \frac{(a-1)(a-2)\cdots}{(n+1)!}(a-n)b^n x^{n+1}.$$

Now $F(a, b; x, y)$ satisfies all the conditions of theorem 4, even when a, b are zero, so that we have

$$y = x + J(a,b\,;x)\,; \quad x = y + J(b,a\,;y),$$

where we easily see that

$$(7) \qquad J(a,b\,;x) = \sum_{n=1}^{\infty} \frac{(a-b)(a-2b)\cdots(a-nb)x^{n+1}}{(n+1)!}.$$

Moreover if $\lambda \neq 0$,

$$\lambda J\left(\lambda a, \lambda b\,; \frac{x}{\lambda}\right) = J(a,b\,;x)$$

and if $ab \neq 0$,

$$1 + ax + aJ(a,b\,;x) = (1+bx)^{a/b}.$$

We can define $J(a, b; x)$ by the series (7) and then prove that $x+J(a, b; x)$ is actually a proper Φ-function. This has been done by O. Jezek.[1] We conclude with a few easily proved but curious properties of the function $J(a, b; x)$. If

$$x = t + J(b,ab\,;t) \qquad \text{and} \qquad y = t + J(a,ab\,;t),$$

then

$$x + J(ab,b\,;x) = y + J(ab,a\,;y)\,;$$
$$y - x = J(a,b\,;x)\,; \quad x - y = J(b,a\,;y).$$

If

$$x = t - J(a,abc\,;t) + J(b,abc\,;t) + J(c,abc\,;t),$$
$$y = t + J(a,abc\,;t) - J(b,abc\,;t) + J(c,abc\,;t),$$
$$z = t + J(a,abc\,;t) + J(b,abc\,;t) - J(c,abc\,;t),$$

then

[1] O. Jezek, *Ueber die Reihenumkehrung*, Wiener Sitzungsberichte Zweite Abteilung, vol. XCIX (1890), pp. 191–203.—See also Whittaker and Watson, *Modern Analysis*, 3rd edition, p. 147, example 14.

$$x - y = J(a, b\,;\, y + z),\ y - x = J(b, a\,;\, x + z)$$
$$y - z = J(b,c\,;\, z + x), \qquad z - y = J(c,b\,;\, y + x),$$
$$z - x = J(c,a\,;\, x + y), \qquad x - z = J(a,c\,;\, z + y),$$
$$J(abc,c\,;\, x + y) = J(abc,a\,;\, y + z) = J(abc,b\,;\, z + x).$$

GENERALIZATIONS IN GEOMETRY AS SEEN IN THE HISTORY OF DEVELOPABLE SURFACES

By FLORIAN CAJORI, University of California

"The mathematicians of the eighteenth century would have been astonished to a high degree, had they been told that there exist developable surfaces which are not ruled surfaces." Perhaps this passage from the pen of Picard[1] surprises many mathematicians even of the present time; it challenges the historian to endeavor to trace the evolution of ideas. The result alluded to is no less surprising to us than was to Euler in the eighteenth century the fact that i^i, where $i = \sqrt{(-1)}$, has a real value. In a letter to Goldbach, Euler showed his interest by computing this value to ten decimal places. Picard's statement is no ess surprising than the declaration about integral numbers made by Galileo in the seventeenth century: "Neither is the number of squares less than the totality of all numbers, nor the latter greater than the former."

Period of Primitive Intuition

Aristotle remarked that "a line by its motion produces a surface."[2] When this line was a straight line, ruled surfaces would result, which clearly included the cone and cylinder. But Aristotle's statement does not necessarily carry the implication that there are ruled surfaces which can be spread out upon a plane. Nevertheless, early students of geometry must have recognized as intuitionally evident the fact that, without stretching or tearing, the curved surface of cylinders and cones could be unbent upon a plane. Explanations of this property are not generally given. We have found the developed surface of a right cone drawn as the sector of a circle, in a practical work on mensuration,[3] without any novelty being claimed for it. In the same treatise the cylinder is described as being "in form of a Rolling stone used in Gardens," an expression conveying the picture of a surface rolled over a plane so that all its points are brought into coincidence with the plane.

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[1] Émile Picard, *La science moderne et son état actuel*, Paris, p. 53.

[2] Aristotle, *De Anima*, I, 4, 409, a4; T. L. Heath's *Thirteen Books of Euclid*, vol. 1, 2nd edition (1926), p. 170.

[3] William Hawney, *The Complete Measurer*, ninth Edition (1755), p. 159. See also p. 154. (First edition, London, 1717).

The French writer on stone-cutting, A. F. Frézier,[1] considered the application to the plane of curved surfaces of the *oblique* circular cylinder and cone. He considered certain skew surfaces but did not distinguish accurately between developable surfaces and general ruled surfaces.

First Generalization—Infinitesimal Analysis used in the Treatment of Developable Surfaces

The first critical studies of developable surfaces were made by Leonhard Euler and Gaspard Monge. The two investigators approached the subject about the same time, but Euler's paper received earlier publication, in 1772. It is noteworthy that at this time Euler was blind. The title of his paper, "On solids whose surface may be spread out upon a plane,"[2] shows that surfaces were not yet looked upon as distinct entities, but as boundaries of solids. Euler asks himself the question, are there other solids than the cylinder and cone, whose surfaces can be unfolded upon a plane? (quorum superficiem itidem in planum explicare liceat nec ne?). He gives the topic three different treatments. The first is purely analytic, the second is geometric and trigonometric, the third considers surfaces formed by rays of light enveloping the shadow cast by an opaque body illuminated by a luminous disk.

In the first method he assumes that an infinitesimal right triangle whose right vertex is x, y, z passes into a congruent right triangle in the plane whose right vertex is t and u. Euler tacitly assumes, as is to be expected of eighteenth century mathematicians, that the differential and integral calculus can be applied to the problem in hand, that, in other words, there exist in the geometry of these surfaces limiting values called derivatives. He takes the vertices of the right triangle in the plane to be t, u; $t+dt, u$; $t, u+du$. The coordinates of the vertices of the corresponding right triangle in the surface are x, y, z; $x+ldt$, $y+mdt$, $z+ndt$; $x+\lambda du$, $y+du$, $z+\nu du$. Euler obtains six equations which on his assumptions are the necessary and sufficient analytical conditions that the surface be developable upon the plane, viz.,

$$(dl/du) = (d\lambda/dt), \quad (dm/du) = (d\mu/dt),$$

$$(dn/du) = (d\nu/dt), \quad l^2 + m^2 + n^2 = 1, \quad \lambda^2 + \mu^2 + \nu^2 = 1, \quad l\lambda + m\mu + n\nu = 0,$$

where the parentheses indicate partial derivatives, and where l, m, n, λ, μ, ν are certain unknown functions of t and u, the determination of which is a "problem by itself considered most difficult, but whose solution will be shown further on in most elegant manner."

This solution is accomplished in the course of the second or *geometric* treat-

[1] Amédée François Frézier, *La théorie et pratique de la coupe des pierres et des bois*, Strasbourg, 1737–39. Our information on Frézier is drawn from C. Wiener, *Lehrbuch der Darstellenden Geometrie*, vol. 1, Leipzig, 1884, p. 23, 24.

[2] L. Euler, *De solidis qvorvm svperficiem in planvm explicare vicet*, Novi commentarii academiae scientiarvm imperiatis Petropolitanae, Tom. XVI, pro anno 1771, Petropoli, 1772, p. 3–34.

ment of developable surfaces other than cones and cylinders. Euler takes a sheet of paper and draws non-intersecting straight lines upon it which are not parallel nor all directed to a common point. He considers the possibility of folding the sheet along these straight lines so as to form a surface. He says: "In this sheet it is possible to draw straight lines Aa, Bb, Cc, etc. at pleasure, such that none of them are parallel nor all converging to a fixed point, provided that they nowhere intersect each other, as shown in Fig. 1. However that sheet is bent along the straight lines, it is always possible to conceive of a solid which fits that bent sheet. From this it follows that besides prismatic and pyramidal bodies there are any number of other kinds of bodies which may be covered in this manner by that sheet, and whose surface may accordingly be unfolded upon a plane."

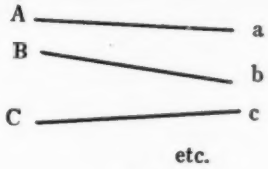

FIG. 1.

We quote further:

"Let us now increase to infinity [the number of] those lines Aa, Bb, Cc, etc. so that our solid acquires a surface everywhere curved, as our problem demands according to the law of continuity. And now it appears at once, that the surface of such bodies should be so constituted that from any point in it at least one straight line may be drawn which lies wholly on this surface; although this condition alone does not exhaust the requirements of our problem, for it is necessary also that any two proximate straight lines lie in the same plane and therefore meet unless they are parallel."

Euler explains that the points of intersection of pairs of neighboring straight lines form on the surface a twisted curve of double curvature, and that, conversely, given any twisted curve, one may derive from it a developable surface formed by the successive tangents of that curve. He establishes the analytic relation between the coordinates t, u, v of a point on the given twisted curve and the coordinates x, y, z of a corresponding point on the resulting developable surface. He connects the formulas thus obtained with the results of his first investigation involving the undetermined functions l, m, n, λ, μ, ν, by showing that $dl:d\lambda = dm:d\mu = dn:d\nu = -\cos\omega:\sin\omega$, where ω is an angle in his geometric figure. "Accordingly, if we examine carefully these things, we discern certain paths along which we may run down the direct solution of that most difficult problem."

About the same time, and independently of Euler, the subject of developable surfaces was investigated by Gaspard Monge, the creator of descriptive geometry. His earliest publication on such surfaces appeared at Paris in 1785; he discussed them repeatedly in later writings. Monge's treatment is less analyti-

cal than that of Euler and more nearly the result of direct contemplation of space relations. He starts with a curve of double curvature (twisted curve). Through a point on it he draws a plane perpendicular to a tangent line to the curve at that point. Similarly through a second point on the curve, infinitely near (infiniment proche) to the first, he draws a second plane. The two planes intersect in a line. Drawing in the same manner planes through all "consecutive" points (par tous ses points consecutifs), the lines of intersection of "consecutive" planes form a surface. This surface, he says, enjoys the general property of being developable upon a plane, as do conical and cylindrical surfaces, without overlapping and break of continuity (sans duplicature et sans solution de continuité). Monge observes also that a tangent line to the twisted curve, moving along that curve, generates a developable surface. He then proceeds to determine the equation of a developable surface from the equations of a twisted curve.

The general impression which we received from reading Euler and Monge is that Euler made a more searching inquiry into the nature of surfaces which can be spread out upon a plane. By considering also the opposite operation of taking a sheet of paper and folding it loosely, he came very near to a still broader generalization.

Second Generalization. Imaginary Surfaces.

It might seem desirable, in the treatment of real surfaces, to remain altogether in the realm of the real. But experience has shown this to be impossible of attainment in a general exposition. In the general development of higher geometry as well as the theory of surfaces in particular, the use of imaginary and ideal elements became inevitable. These are creations affording economy of thought and elegance of exposition. Confinement in geometry to real magnitudes definitely located in finite space frequently necessitates the consideration of special cases, which merge beautifully into one general case when imaginaries and elements at infinity are admitted. Two ellipses may intersect in four real points, provided their eccentricities are not zero. In the special case when the eccentricities are zero, the curves are circles and cannot have more than two real points of intersection. For the sake of generality, the geometer speaks of two imaginary *circular points* of the line at infinity. Similarly for spheres. An early leader in this movement toward generalization was the Frenchman Poncelet, in the first decennia of the nineteenth century.

Even in geometrical problems which arose in the eighteenth century, the use of imaginary relations could not be avoided. We illustrate this remark by the case of a minimum surface arising in Lagrange's problem[1] to determine a connected surface bounded by a given closed curve, so that the area of this surface shall be a minimum. Lagrange was led to a partial differential equation of the second order, for which later Monge[2] found a general solution. "Accord-

[1] J. Lagrange, *Essai d'une nouvelle méthode* . . . . , Miscellanea Taurinensia,, vol. 2 (1760–1761); *Oeuvres de Lagrange*, vol. 1 (1877), p. 356.

[2] G. Monge, Mémoires de l'académie r. d. sciences pour 1784, p. 118 et Suppl. p. 536.

ing to the nature of the case," says Lie,[1] "Monge's general integral was affected by imaginaries and consequently for a long time it was possible to find only very particular real minimal surfaces. Bonnet[2] [about 70 years later] was the first to give a general method for finding all real minimal surfaces · · ·." In deriving all real surfaces satisfying Lagrange's problem, Bonnet made liberal use of imaginaries, which, after rendering valiant service, were gotten rid of, by the use of conjugate functions $F(x+iy)$ and $F(x-iy)$.

The use of imaginaries and minimal lines led to unexpectedly new results in developable surfaces. Let the coordinates of the points on a straight line in space be represented by $x=a+At$, $y=b+Bt$, $z=c+Ct$, where a, b, c is a fixed point, and t is a parameter. This straight line passes through the point a, b, c, and through the point $a+A$, $b+B$, $c+C$, as appears by letting t be 0 and 1. There is a point on the line for every value of t. The distance from the point a, b, c to any other point x, y, z on the line is $\sqrt{[(x-a)^2+(y-b)^2+(z-c)^2]}$. If we substitute in this the values for x, y, z, the expression takes the form $t[A^2+B^2+C^2.]^{1/2}$ When A, B, C are all real, and $t>0$, this distance is positive and not zero. In the case of imaginary values of A, B, C, giving rise to imaginary lines, it is possible to have $A^2+B^2+C^2=0$. For example, we may take $A=3$, $B=4$, $C=5i$. Thereby we are led to the so-called "minimal lines," whose lengths are zero for any value of the parameter t. Thus, for real values of A, B, C, one obtains real straight lines whose lengths vary with t; only imaginary straight lines can be minimal, all having their lengths "null."

Suppose next that in place of the fixed point a, b, c, we take the variable point moving along a curve in space, which has its three coordinates $\phi(s)$, $\chi(s)$, $\psi(s)$, where the parameter s is the length of arc on that curve. We then obtain

$$x = \phi(s) + At, \quad y = \chi(s) + Bt, \quad z = \psi(s) + Ct$$

as the equations of a cylinder. If, as before, A, B, C are made to satisfy the equation $A^2+B^2+C^2=0$, then the cylinder is imaginary and composed of minimal lines. It has been shown that to every point in this imaginary cylinder there corresponds a point in the plane such that corresponding arcs on the cylinder and in the plane are of equal length. Thus, it appears that *the imaginary cylinder is developable upon a plane*[3]—a result which would have astonished most of the eighteenth century geometers.

Minimal lines are of course a special type of "minimal curves," a name introduced by Sophus Lie.[4] But the analytical expressions representing minimal curves occur much earlier, in the writings of Monge, Legendre, Enneper and Weierstrass. It has been found that the discussion of geodesics upon a surface is much simplified by referring the surface to null lines as parametric curves.[5]

[1] S. Lie, Mathematische Annalen, vol. 14 (1879), p. 331.

[2] Ossian Bonnet, Comptes Rendus, vol. 37, Paris (1853), pp. 529–533.

[3] Georg Scheffers, *Einführung in die Theorie der Curven*, vol. 1 (1901), p. 288; See also V. Kommerell u. K. Kommerell, *Raumkurven u. Flächen*, vol. 2 (1903), p. 183.

[4] S. Lie, Mathematische Annalen, vol. 14 (1879), p. 337.

[5] A. R. Forsyth, *Lectures on Differential Geometry*, Cambridge (1912), p. 76.

Third Generalization. Transcending the Limitations of Infinitesimal Analysis

The theory of surfaces was enriched dring the nineteenth century by the great researches of Gauss, Weierstrass, Bonnet, Darboux and others, which are greatly admired for the generality of the results. Nevertheless, the close of that century brought the astonishing revelation that there existed surfaces applicable to the plane which are not ruled surfaces. The new generalization, due to H. Lebesgue, is no less uncanny than is Peano's famous "space-filling curve." We give in his own words[1] the description of one of Lebesgue's surfaces as that is not ruled, yet is applicable to the plane: "To obtain [such] surfaces that are not ruled I take an analytic developable one and upon it an analytic curve C, not geodesic. One knows that there exists another developable surface passing through C, such that one can make the two developables applicable to a plane in such a manner that to each point of C, whether considered as belonging to the one or to the other of the two surfaces, there corresponds one and the same point in the plane. The curve C divides the first developable in two pieces, A, B, also the second into two pieces A', B'.

"Two of the four surfaces (A, A'), (A, B'), (B, A'), (B, B') are applicable to the plane without tearing or duplication and are indeed such that one can detach from them a finite piece enjoying the above property and containing an arc of C. This C is then a singular line. As before, one may pass from this one singularity to an infinite number of singularities and one thus obtains surfaces applicable to the plane, yet not containing any segment of a straight line."

In this compact statement it is hard to picture the effect of an infinite number of repetitions of the process described.

Lebesgue gives a second mode of variation, the unlimited repetition of which leads to an unruled surface that is applicable to the plane: "In general, if $y=f(x)$ is a curve, limited in its variation so that its total variation from x_0 to x_1 is $K|x_0-x_1|$, K being constant, the surface generated by this curve when rotated about Oy, is applicable to the plane."

It is worth while to supplement these passages by the comment of Picard:[2] "According to general practice, we suppose in the preceding analysis, as in all infinitesimal geometry of curves and surfaces, the existence of derivatives which we need in the calculus. It may seem premature to entertain a theory of surfaces in which one does not make such hypotheses. However, a curious result has been pointed out by Mr. Lebesgue (*Comptes Rendus*, 1899 and thesis); according to which one may, by the aid of continuous functions, obtain surfaces corresponding to a plane, of such sort that every rectifiable line of the plane has a corresponding rectifiable line of the same length of the surface, nevertheless the surfaces obtained are no longer ruled. If one takes a sheet of paper, and crumples it by hand, one obtains a surface applicable to the plane and made up of a

[1] Comptes Rendus, vol. 128 (1899), p. 1502–1505; a second article of Lebesgue on this topic is found in Annali di matematica (Brioschi), 3rd series, vol. 7, Milan (1902), p. 324.

[2] Émile Picard, *Traité d'analyse*, 3rd edition, vol. 1, Paris (1822), p. 555, foot-note.

finite number of pieces of developable surfaces, joined two and two by lines, along which they form a certain angle. If one imagines that the pieces become infinitely small, the crumpling being pushed everywhere to the limit, one may arrive at the conception of surfaces applicable to the plane and yet not developable [the envelope of a family of planes of one parameter] and not ruled."

The crumpling of a sheet of paper reminds one of Euler's process of fold-along straight lines. Euler's procedure was regular, systematic—too much so to yield the results of Lebesgue.

A RELATION BETWEEN POLAR CONICS AND OSCULANT CONICS OF A NODAL CUBIC

By FRANC C. EARHART

Brill,[1] in developing the theory of involutions on rational curves, considered that all projective properties of a curve are given by properties of certain involutions of groups of points on this curve. In the geometric development of this theory use has been made of certain covariant systems of curves of lower order. Study[2] and Jolles[3] have treated these systems analytically and the latter has given to them the name "osculants."[4]

The osculants of rational curves are defined as follows:[5] If the parametric point equations of a rational curve are

$$x_i = f_i(t,\tau), \quad i = 1,2,3, \tag{1}$$

where f_i are binary forms of order n in the homogeneous parameter t/τ, then the equations

$$x_i = \left(t_1 \frac{\partial}{\partial t} + \tau_1 \frac{\partial}{\partial \tau}\right) f_i, \tag{2}$$

obtained by taking the first polars of f_i with respect to (t_1,τ_1), represent the first osculant of (1) at the point t_1/τ_1. Likewise polarizing (2) with respect to t_2/τ_2, we obtain

$$x_i = \left(t_2 \frac{\partial}{\partial t} + \tau_2 \frac{\partial}{\partial \tau}\right)\left(t_1 \frac{\partial}{\partial t} + \tau_1 \frac{\partial}{\partial \tau}\right) f_i,$$

which is the first osculant of (2) and a second osculant of (1).

The process may be continued until the f_i's are completely polarized.

Osculants at the point t_1 touch the curve there; all first osculants touch the

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[1] Mathematische Annalen, vol. 20 (1882), p. 335.

[2] *Über die Raumcurve vierter Ordnung zweiter Art*, Sitzungsberichte der Königlichen Sächsische Gesellschaft der Wissenschaften, vol. 38 (1886).

[3] *Theorie der Osculanten, etc.*, Habilitationsschrift, Aachen (1886).

[4] Journal für die reine und angewandte Mathematik, vol. 101 (1887), p. 300.

[5] Winger, *Projective Geometry* (1923) pp. 386–387.

stationary tangents of the curve, and the $(n-1)$st osculant is the tangent to the curve.[1]

It is the purpose of this paper to study a relation between the first polars and the first osculants of rational cubics.

The linear polar of a point on a curve is the tangent at the point and thus coincides with the linear osculant at that point. But, in general, the polar conic of a point on a cubic is not identical with the osculant conic at that point.

A canonical form for the parametric equations of the rational nodal cubic is[2]

$$x = 3t^2, \qquad y = 3t, \qquad z = t^3 + 1, \tag{3}$$

the flex parameters being -1, $-\omega$, $-\omega^2$, the nodal parameter 0, and the line $z=0$ the line of flexes, or in the homogeneous form:

$$x = 3t^2\tau, \qquad y = 3t\tau^2, \qquad z = t^3 + \tau^3. \tag{4}$$

Then the osculant curve at the point t_1 is given by the equations:

$$x = t^2 + 2t_1t, \qquad y = 2t + t_1, \qquad z = t_1t^2 + 1. \tag{5}$$

Eliminating t in (3), we obtain the equation of the cubic: $x^3+y^3-3xyz=0$. The polar conic of x_1, y_1, z_1 is

$$x_1(x^2 - yz) + y_1(y^2 - xz) - z_1xy = 0,$$

or, substituting for x_1, y_1, z_1, their respective values from (3),

$$3t_1^2(x^2 - yz) + 3t_1(y^2 - xz) - (t_1^3 + 1)xy = 0.$$

To determine the intersections of the osculant conic and the polar conic, substitute for x, y, z, their respective values from (5) which yields an equation of the fourth degree in t: $t(t-t_1)^2=0$. Therefore $t=\infty$, 0, t_1, t_1 are parameters of the points of intersection of the polar and the osculant conics. t_1 gives the point of contact at which each is known to be tangent to the cubic.

When $t=0$, $x:y:z=0:t_1:1$.

When $t=\infty$, $x:y:z=1:0:t_1$. The equation of the line joining the points $t=0$ and $t=\infty$ is:

$$t_1^2x + y - t_1z = 0.$$

Differentiating with respect to t_1 and eliminating t_1 between the two equations, the envelope is shown to be the conic: $4xy-z^2=0$. This form would suggest the possibility of identifying this envelope with:

1) The polar conic of a special point.
2) The poloconic of a special line.
3) A covariant conic of the cubic.

The absence of the term z^3 in the cubic precludes 1). The poloconic of a line $ax_1+by_1+cz_1=0$, with respect to the cubic $x^3+y^3-3xyz=0$, is found to be[3]

$$(a^2 - 4bc)x^2 + (b^2 - 4ac)y^2 + c^2z^2 - 2(ab + 2c^2)xy - 2bcyz - 2acxy = 0.$$

[1] Thomsen, American Journal of Mathematics, vol. 32 (1910), p. 207.

[2] Winger, *Projective Geometry*, p. 368.

[3] White, *Plane Cubic Curves*, p. 47.

Comparing this with the conic $4xy-z^2=0$,

$$(6) \qquad a^2-4bc=0, \qquad b^2-4ac=0, \qquad bc=0, \qquad ac=0,$$
$$c^2=1, \quad 2(ab+2c^2)=4,$$

$c\neq 0$ since that would imply also $a=b=0$. When $a=b=0$, the conditions in (6) are satisfied. Therefore $4xy-z^2=0$ is the poloconic of the line $z=0$. Hence the following theorem:

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant at a point t_1 of a nodal cubic is the poloconic of the line of flexes.

It is known that the corresponding points A and B on the Hessian are also the points of contact of the two tangents which can be drawn from a point on the curve. Winger has discussed the envelope of the line joining two such points in his study of involutions on the rational cubic.[1] This envelope, the Cayleyan of the original cubic, is a tri-tangent conic of the Hessian which Winger has called "the involution conic of the node." Hence the above theorem may be stated also in the following forms:

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant of a point t_1 of a nodal cubic is the Cayleyan of the cubic.

The envelope of the line joining the two points of intersection (not the point of tangency) of the first polar and the first osculant of a point t_1 of a nodal cubic is the involution conic of the node of the Hessian.

Since the points of intersection of the osculant and polar conics of the cuspidal cubic are absorbed by the point t_1 and the cusp, there is no envelope problem.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. A GRAPHICAL DERIVATION OF CRAMER'S RULE

By J. P. BALLANTINE, University of Washington

Consider first the case of two equations in two unknowns:

$$(1) \qquad a_{11}x_1+a_{12}x_2=a_{13},$$
$$a_{21}x_1+a_{22}x_2=a_{23}.$$

[1] Winger, *Projective Geometry*, p. 383.

Considering the vectors $A_1=(a_{11}, a_{21})$, $A_2=(a_{12}, a_{22})$, $A_3=(a_{13}, a_{23})$, the solution of (1) is equivalent to the determination of two numbers x_1, x_2, such that

$$A_1x_1 + A_2x_2 = A_3. \tag{2}$$

Let the vectors A_1, A_2, A_3 be as illustrated in Fig. 1. The parallelograms A_1A_2 and A_3A_2 are completed and denoted by P_{12} and P_{32}. The vector A_1x_1 is determined by the following two conditions: (1) It is a multiple of A_1; and

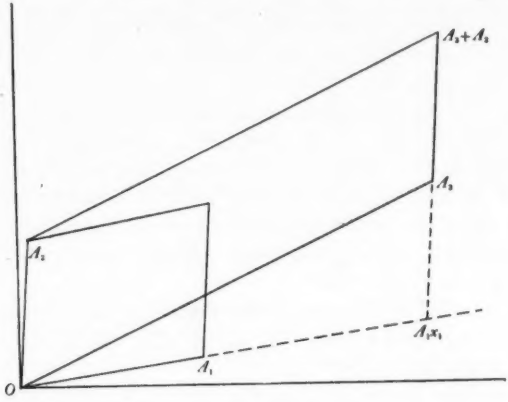


FIG. 1.

(2) if a proper multiple of A_2 is added to it, the sum is A_3. The first of these conditions restricts A_1x_1 to the line from O to A_1, and the second restricts it to the line from A_3 to A_3+A_2.

The value of x_1 is thereby determined as the ratio A_1x_1/A_1. This in turn is seen to be equal to the ratio between the altitudes of P_{32} and P_{12}, the altitude in both cases being taken on the common side, namely the line of the vector A_2. This in turn is seen to be equal to the ratio between the areas of P_{32} and P_{12}. Since these areas are given by certain two row determinants, we have the familiar formula for x_1 in terms of determinants.

The above derivation of Cramer's rule for the case $n=2$ appears at first to be dependent on the fact that $n=2$, and hence not capable of extension. For $n\geqq 3$, one could not plot the vectors except in perspective, and then the ratio of A_1x_1/A_1 would be affected. A closer examination, however, of the above derivation shows that the diagram did not enter into the proof. Let us illustrate by considering the case $n=3$.

The system of equations to be solved is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= a_{34}. \end{aligned} \tag{3}$$

The vectors A_1, A_2, A_3 are defined as (a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33}),

namely the columns of the matrix of coefficients, and $A_4=(a_{14}, a_{24}, a_{34})$, the set of right hand members. Instead of (2), we have

$$A_1x_1 + A_2x_2 + A_3x_3 = A_4. \tag{4}$$

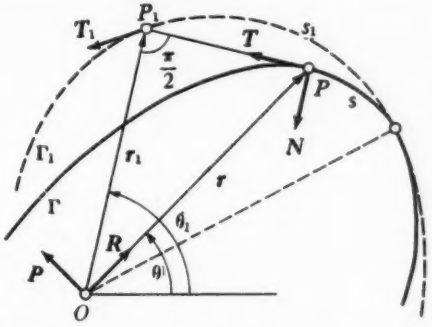
The parallelopiped obtained by completing the vectors A_i, A_j, A_k shall be denoted by P_{ijk}. The vector A_1x_1 is determined by the following two conditions: (1) It is a multiple of A_1, and (2) if a proper linear combination of A_2 and A_3 is added to it, the sum is A_4. The first of these conditions restricts A_1x_1 to the line from O to A_1, and the second restricts it to a plane, namely the plane passing through the point A_4, and parallel to the lines of the vectors A_2 and A_3. Excluding the case in which Cramer's rule does not hold, the determinant of the coefficients does not vanish, the area of P_{123} is not zero, and A_1 is not a linear combination of A_2 and A_3. Hence the above line is not parallel to the above plane, and they intersect at a point, thus uniquely determining the vector A_1x_1.

The value of x_1 is thereby determined as the ratio A_1x_1/A_1. This in turn is seen to be equal to the ratio between the altitudes of P_{423} and P_{123}, the altitudes in both cases being taken as the altitudes on the common face, namely the face whose two edges are the lines of the vectors A_2A_3. This in turn is seen to be equal to the ratio between the volumes of P_{423} and P_{123}. Since these volumes are equal to certain well known three row determinants, the value of x_1 has been obtained, and Cramer's rule derived.

II. A Note on Pedals of Plane Polar Curves

By H. K. Justice, University of Cincinnati

The usual method of obtaining the polar equation of the first positive pedal of any plane polar curve, with regard to the pole, requires the elimination of three quantities from four equations.



The purpose of this discussion is to present a rather interesting special method of deducing parametric equations of the pedal curve by expressing each of the coördinates of a point on the pedal as a function of either of the coördinates of the corresponding point on the original curve, through integrals involving the radius of curvature of the latter.

Let the point $P(r, \theta)$ traverse the arc, s, of the given curve, Γ, in such a way that the radius vector, $\boldsymbol{r}=OP$, rotates counterclockwise, in the positive sense determined by a unit binormal vector, $\boldsymbol{B}$, directed toward the reader. This establishes a positive direction for the unit tangent vector, $\boldsymbol{T}=d\boldsymbol{r}/ds$. Let $\boldsymbol{R}$ be a unit vector in the direction of $\boldsymbol{r}$, and define other unit vectors $\boldsymbol{N}$ and $\boldsymbol{P}$, according to the right hand rule, by the equations,

$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}, \quad \boldsymbol{P} = \boldsymbol{B} \times \boldsymbol{R}.$$

Designate by κ and ρ the curvature and radius of curvature of Γ, at P. Characterize, by the subscript 1, the symbols associated with the pedal, Γ_1, at the corresponding point $P_1(r_1, \theta_1)$. Evidently $\boldsymbol{r}_1=\boldsymbol{r}\cdot\boldsymbol{NN}$. Also[1]

$$\boldsymbol{T}_1 = \frac{d\boldsymbol{r}_1}{ds_1} = \frac{d\boldsymbol{r}_1}{ds}\frac{ds}{ds_1} = \frac{d}{ds}(\boldsymbol{r}\cdot\boldsymbol{NN})\frac{ds}{ds_1},$$

which, by virtue of Frenet's formula, $d\boldsymbol{N}/ds=-\kappa\boldsymbol{T}$, becomes

$$\boldsymbol{T}_1 = -\kappa(\boldsymbol{r}\cdot\boldsymbol{NT}+\boldsymbol{r}\cdot\boldsymbol{TN})\frac{ds}{ds_1}.$$

Differentiating the obvious identity, $\boldsymbol{r}_1=-r_1\boldsymbol{N}$, we obtain

$$\boldsymbol{T}_1 = -\frac{dr_1}{ds_1}\boldsymbol{N} - r_1\frac{d\boldsymbol{N}}{d\theta_1}\frac{d\theta_1}{ds_1} = -\frac{dr_1}{ds_1}\boldsymbol{N} + r_1\frac{d\theta_1}{ds_1}\boldsymbol{T}.$$

Hence

$$-\frac{dr_1}{ds_1}\boldsymbol{N} + r_1\frac{d\theta_1}{ds_1}\boldsymbol{T} = -\kappa(\boldsymbol{r}\cdot\boldsymbol{NT}+\boldsymbol{r}\cdot\boldsymbol{TN})\frac{ds}{ds_1}. \tag{1}$$

Taking components along $\boldsymbol{T}$, we see that

$$r_1\frac{d\theta_1}{ds_1} = -\kappa r\boldsymbol{R}\cdot\boldsymbol{N}\frac{ds}{ds_1} = -\kappa r\boldsymbol{R}\times\boldsymbol{B}\cdot\boldsymbol{T}\frac{ds}{ds_1} = \kappa r\boldsymbol{P}\cdot\boldsymbol{T}\frac{ds}{ds_1},$$

from which it follows, by virtue of the equation,

$$\boldsymbol{T} = r\boldsymbol{P}\frac{d\theta}{ds} + \frac{dr}{ds}\boldsymbol{R}, \tag{2}$$

obtained by differentiating the identity, $\boldsymbol{r}=r\boldsymbol{R}$, that

$$d\theta_1 = \frac{\kappa r^2}{r_1}d\theta = \frac{\kappa r^2}{r_1 r'(\theta)}dr. \tag{3}$$

Similarly, resolving (1) in the direction of $\boldsymbol{N}$, we find that

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[1] The writer acknowledges indebtedness to Professor Louis Brand, who has previously employed similar vector methods in the solution of other problems in differential geometry.

$$\frac{dr_1}{ds_1} = \kappa r \boldsymbol{R} \cdot \boldsymbol{T} \frac{ds}{ds_1},$$

which, with (2), yields the known result,

$$dr_1 = \kappa r dr = \kappa r r'(\theta) d\theta. \tag{4}$$

Thus the equations,

$$r_1 + C_1 = \int \frac{r}{\rho} dr = \int \frac{r}{\rho} r'(\theta) d\theta, \tag{5}$$

$$\theta_1 + C_2 = \int \frac{r^2}{\rho r_1 r'(\theta)} dr = \int \frac{r^2}{\rho r_1} d\theta, \tag{6}$$

obtained by integrating (4) and (3) respectively, express the coördinates, (r_1, θ_1), of a point on the pedal curve, as functions of either r, or θ, as a parameter. The constants of integration, C_1 and C_2, may be determined from the condition that the points (r, θ) and (r_1, θ_1) obviously coincide for ordinary extremes of r.

Example: Find the pedal of the curve $r^a = b^a \cos a\theta$, whose radius of curvature is $\rho = b^a r^{1-a}/(a+1)$.

From (5) and (6),

$$r_1 + C_1 = \int \frac{r dr}{\rho} = \frac{a+1}{b^a} \int r^a dr = \frac{r^{a+1}}{b^a} = b \cos^{(a+1)/a} a\theta,$$

$$\theta_1 + C_2 = \int \frac{r^2 d\theta}{\rho r_1} = \int \frac{(a+1) r^2 b^a}{b^a r^{1-a} r^{a+1}} d\theta = (a+1) \int d\theta = (a+1)\theta.$$

When $\theta = 0$, r has the maximum value b. Hence when $\theta = 0$, $r_1 = r = b$, and $\theta_1 = \theta = 0$. Therefore $C_1 = C_2 = 0$. By eliminating θ from the above equations, we deduce the polar equation of the pedal curve in the form,

$$r_1^{a/(a+1)} = b^{a/(a+1)} \cos \frac{a\theta_1}{a+1},$$

which obviously represents a member of the original family, and is obtainable from the equation of the latter by changing a to $a/(a+1)$. Interesting special cases result from specific numerical substitutions for a.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Ames, J. S. and *Murnaghan, F. D.* Theoretical Mechanics. An Introduction to Mathematical Physics. Boston, Ginn and Co., 1929. ix+462 pages. $5.00.

Rietz, H. L. and *Crathorne, A. R.* College Algebra, Third Edition. New York, Henry Holt and Co., 1929.

"The book has been thoroughly revised again. · · · The exercises and problems have been [almost] completely changed. · · · The chapter on probability is new. · · · A new chapter on compound interest and annuities has been added."

Ford, Lester R. Automorphic Functions. New York, The McGraw-Hill Book Co., 1929. xii+334 pages. $4.50.

Coble, Arthur B. Algebraic Geometry and Theta Functions. American Mathematical Society Colloquium Publications, Vol. X. New York, 1929. viii+282 pages.

"An amplification of the Colloquium lectures delivered at Amherst in September, 1928 under the title, "The determination of the tri-tangent planes of the space sextic of genus four."

Schorling, Raleigh and *Clark, John R.* Modern Mathematics, New Edition. Yonkers, N. Y., The World Book Co., 1929. Seventh School year, xiv+274 pages; Eighth School Year, xiv+306 pages.

Campbell, J. W. An Introduction to Mechanics. Boston, Houghton, Mifflin Co., 1929. xiv+384 pages. $3.50.

Reynolds, Joseph B. Elementary Mechanics. New York, Prentice-Hall, Inc., 1929. viii+250 pages. $2.50.

The first of these two texts on mechanics presupposes elementary calculus, while the second is based only on secondary school algebra, geometry, and trigonometry.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Numerische Infinitesimalrechnung. By Martin Lindow. Ferd. Dümmlers, Berlin, 1928. 176 pages.

Interpolation. By J. F. Steffensen. The Williams and Wilkins Company, Baltimore. Md., 1927. 248 pages. $8.00.

Mathematical books belong to a number of different categories. Among them are monographs of original research; expositions of work scattered throughout the literature, bringing into available form and into a given language the original work of others; and, appearing at rare intervals, books doing both, making available what is already known, filling in the gaps and extending the theory.

Numerische Infinitesimalrechnung is of the second type, a useful, well arranged, clearly written tract on interpolation, in German. *Interpolation* by Professor J. F. Steffensen is in the last class, being written so clearly that one does not realize while first reading it how much novel material the book contains.

Each of these books has the virtue of having an honest introduction that gives the key to the subject matter and mode of approach.

Dr. Lindow says in his foreword: "Pure mathematics stresses absolute rigor and generality. But this is often unsuited to the practical examinations of a concrete case. That which is of the greatest use for the physicist, chemist, and engineer requires no proof." Dr. Lindow does not pretend to give a rigorous or novel mathematical treatment, but by a multiplicity of examples and clear explanations of the use of formulas, supplemented by tables of coefficients appearing in formulas frequently used in interpolation, he tries to be of the utmost use to the computer. The book should be especially commended for these tables. The general scope is given by the following headings: interpolation, numerical differentiation, numerical integration and numerical handling of differential equations. The scope of *Interpolation* is nearly coextensive except for the chapter on the "Calculus of Symbols." The topics discussed are interpolation, numerical differentiation, construction of tables, summation processes, mechanical quadrature, numerical integration of differential equations, interpolations with several variables, mechanical quadrature and the calculus of symbols.

Professor Steffensen, in the preface and introduction of *Interpolation*, gives the following outline of the purpose of the book:

"Formulas and methods are developed on the assumption that the function under consideration is a polynomial, and thereafter applied to functions which are certainly not polynomials." "Attempts have been made, now and then, to present the subject of interpolation, adopting the point of view that only such approximative formulas are to be included for which it is possible to derive a remainder-term simple enough to permit the calculation of limits to the error involved in the formula." "The number of formulas with workable remainder-terms has lately increased so much, that although further development is still possible and desirable, a fresh attempt should be made at writing a text-book on the aforesaid lines." "I wish it to be understood that the book is meant as a text-book, and not as a hand-book or encyclopedia on the subject. To carry through with consistency the point of view which appears to me to be the only tenable one, has been my principal aim." "The mathematical equipment required in order to master the book is very small. A knowledge of the first principles of the differential and integral calculus should be sufficient." "In practice, it is a very general custom to derive formulas of interpolation on the assumption that the function with which we have to deal is a polynomial of a certain degree. . . . If it is applied to a polynomial of higher degree or to a function which is not a polynomial, nothing whatever is known about the accuracy obtained. . . . If we have to deal with a numerical calculation, it is not sufficient to know that an approximation is obtainable; what we want to know, is how close is the approximation actually obtained."

It is a pleasure to say that after stating his intention, Professor Steffensen hews to the line. The distinction between interpolation when a remainder term

is known and interpolation when the result is merely a guess is emphasized throughout; for instance, on page 53: "The result of an interpolation under such circumstances must be considered as an hypothesis, and not as a mathematically proved fact."

The consistency with which this purpose is carried out is seen from the fact that such formulas as Newton's, Stirling's, Everett's, Bessel's and Gauss's and Lagrange's interpolation formulas, the formulas of numerical differentiation, inverse interpolation and mechanical quadrature, and such formulas as Laplace's, Gauss's, Euler's, and Lubbock's and Woolhouse's summation formulas are all given with remainder terms. A great effort has been made to make these remainder terms as applicable and simple as possible. This has met with striking success. Even the formulas of interpolation in more than one variable, of mechanical quadrature, and of numerical integration of differential equations are supplied with remainder terms.

As a striking warning to our intuition the following example is developed: Consider $1/(1+x^2)$ in the interval $-5 \leqq x \leqq 5$; divide the interval into $5v$ equal intervals, where $v=4k-1$. It would seem to one's intuition that a polynomial going through the end points of these intervals would approximate the function more and more closely as k increases. However, it is found that the maximum error becomes infinite exponentially with increasing k.

The whole book is written on a different plan from most discussions of interpolation, being a worthy continuation of Markoff's work. It can be read with pleasure by the mathematician as the proofs are elegant and rigorous, the hypotheses are clearly stated and in no place is intuition used instead of brains, as in many works on applied mathematics. Moreover, the forbidding symbolism usual in the discussion of this subject is minimized. The problems are well chosen to illustrate the theory, though perhaps those in Dr. Lindow's book more completely indicate the difficulties the computer may meet.

It would be unfair to *Interpolation* not to speak of its physical aspects. It is printed on paper which is pleasant to handle and in clear, attractive type.

Rarely does the reviewer read a book which is mathematically as completely satisfying as *Interpolation* and at the same time one which a good senior could read, except in rare spots, with only moderate difficulty and with great profit. No one who has much to do with interpolation either theoretical or applied could afford to do without Professor Steffensen's book, while *Numerische Infinitesimalrechnung* will prove of use to many computers.

M. H. INGRAHAM

Mengenlehre. By E. Kamke. Berlin, Walter de Gruyter & Co., 1928. 159 pages.

The subject matter of this excellent little book is the general theory of aggregates, with particular reference to transfinite numbers. It is divided into four chapters, as follows: I, the definition of such terms as sum, divisor, etc., and the elementary theorems on enumerable sets; II, the usual theorems on

cardinal numbers and their arithmetic; III, order-types in general, including the identification of the number-continuum with its order-type; IV, the ordinal numbers and the associated aleph numbers. The book is rich in illustrations and applications to the point-set theory. The author follows the traditional treatment of the subject, using Cantor's definition of an aggregate and ignoring the controversy over the axiom of choice; at the end, however, he gives a brief account of the well-known paradoxes. The book covers an astonishing amount of ground for its size, and is clearly and simply written. It should be very useful as an introduction to the subject and it is just the thing for the beginner in the theory of functions of real variables, who needs the commonly accepted theorems as tools, but does not desire to plough through the difficult philosophical questions involved. Harassed directors of undergraduate mathematical clubs may also find something to interest them.

WALLACE A. WILSON

Six-Place Tables, a selection of tables of squares, cubes, square roots, cube roots, fifth roots and powers, circumferences and areas of circles, common logarithms of numbers and of the trigonometric functions, the natural trigonometric functions, natural logarithms, exponential and hyperbolic functions, and integrals. With explanatory notes by Edward S. Allen. Third Edition. McGraw-Hill Book Company, New York, 1929. $1.50.

The scope of this little book—found by experiment to be almost literally a vest-pocket edition—is well indicated by the complete title, as given above. There are six-place tables of logarithms of numbers, and of the trigonometric functions and their logarithms. Natural logarithms occupy one page, and exponential and hyperbolic functions two pages. Table I, occupying sixteen pages, gives certain functions of all integers up to 1000, namely the square, cube, square root, cube root, reciprocal, and circumference and area of circle having the given radius. Table III, curiously enough, seems superfluous as well as incorrect. It tabulates circumferences and areas of circles for radii increasing by eighths of a unit up to 100; and the entries, given to six significant figures, are based on the value 3.1416 for π, and therefore are in many cases in error in the last figure. The same data as given in table I are correctly determined, circumferences to six significant figures and areas to eight.

There is a list of about 150 integrals, with a simple summary of the method of partial fractions.

R. A. J.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3389. *Proposed by J. V. Uspensky.*

Let the function $F(s)$ be defined as follows:

$$F(s) = \frac{1}{(n-1)!} \sum_{\nu=0}^{\nu=s} (-1)^{\nu} \frac{n(n-1)\cdots(n-\nu+1)}{\nu!} (s-\nu)^{n-1},$$

n being a positive integer and s a positive variable. Show that for every n which is very large the following equation holds:

$$\int_0^n s^{-1}F(s)ds = 2n^{-1} + n^{-2}\theta_n,$$

where θ_n remains bounded when n increases indefinitely.

3390. *Proposed by Paul Wernicke, Washington, D. C.*

With the altitudes of a triangle $t_i = A_iB_iC_i$ as sides construct a consecutive triangle t_{i+1} in the series of $t_i(\cdots -1, 0, 1, \cdots)$. Compare the areas of the triangles t_i in the series. Under what condition does the construction become impossible?

3391. *Proposed by Otto Dunkel, Washington University.*

The ratio of the shortest diagonal to a side of an ordinary regular polygon of 19 sides satisfies an equation of the 9th degree with rational coefficients. The remaining roots are the corresponding ratios with alternately plus and minus signs for the remaining regular polygons of star form. By a known theorem the roots of this equation may be obtained by solving first a cubic with rational coefficients and then solving three other cubics whose coefficients are rational functions of the roots of the first cubic. Derive with as little computation as possible a set of such equations.

3392. *Proposed by V. M. Spunar, Chicago, Ill.*

Two points M and N are taken on the sides AB and AC, respectively, of the triangle ABC; and then the point P is taken on the line MN. If these points are chosen so that $BM/MA = AN/NC = MP/PN$, find the locus of P.

3393. *Proposed by Emma M. Gibson, Central High School, Springfield, Mo.*

Suppose a solid is altered so that straight lines remain straight lines and parallel lines parallel. Show that a parallelogram remains a parallelogram, a straight line is stretched to the same extent at all points, parallel straight lines are equally stretched, and an ellipse becomes an ellipse. I.C.S. 1902.

SOLUTIONS

3342 [1928, 445]. *Proposed by R. Goormaghtigh, La Louvière, Belgium.*

Let $\alpha, \beta, \gamma, \delta$ be the points where the straight lines AP, BP, CP, DP, meet the faces of the tetrahedron $ABCD$; the perpendiculars dropped from the vertices A, B, C, D on the lines joining respectively $\alpha, \beta, \gamma, \delta$ to the orthocentre H meet the corresponding faces of the tetrahedron $ABCD$ in four coplanar points. The plane passing through these four points is perpendicular to PH.

Note by the Editors: This problem is a generalization of Problem 3228 [1926, 525 and 1928, 42]. See also the solution of 3258 [1928, 210].

Solution by Nathan Altshiller-Court, University of Oklahoma.

The orthocenter H of the given tetrahedron $ABCD$ is the center of the sphere (H) with respect to which the tetrahedron is self polar. The polar plane of α with respect to (H) passes through A and is perpendicular to αH, hence this plane contains the perpendicular line $A\alpha'$ dropped from A to the line αH. The trace α' of $A\alpha'$ in the plane BCD is thus conjugate to α with respect to (H).

The polar plane of α' with respect to (H) contains the two points A, α conjugate to α' with respect to (H); hence this plane contains also the point P, which is collinear with A and α. Similarly for the points β', γ', δ' analogous to α'. Consequently the four points $\alpha', \beta', \gamma', \delta'$ lie in the same plane, namely, the polar plane of P with respect to (H), which plane is therefore perpendicular to PH.

Note: By a strange coincidence the proposer and the present writer have simultaneously and quite independently of each other stated and proved the proposition in the plane of which the proposition under discussion is an immediate extension. The plane proposition appeared in the April, 1928, issues of "Mathesis" (p. 173) and of this "Monthly" (p. 210).

Also solved by the Proposer.

Note by the Editors: In the proof above, the sphere (H) is not real if H lies within the tetrahedron. In this case the device used in the note [1928, 211] may be employed to complete the proof.

3343 [1928, 446]. *Proposed by J. V. Uspensky.*

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{2n\pi}^{\infty} \frac{\sin z}{z} dz = \pi - \frac{\pi}{2} \log 2\pi$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{n\pi}^{\infty} \frac{\sin z}{z} dz = \frac{\pi}{2} - \frac{\pi}{2} \log \pi.$$

Solution by Harry Langman, Arverne, L. I.

The first sum may be written

$$A = \sum_{n=1}^{\infty} \frac{1}{n} \int_{2\pi}^{\infty} \frac{\sin nz}{z} dz = \sum_{n=1}^{\infty} \frac{1}{n} \int_{\pi}^{\infty} \frac{(-1)^n \sin nz}{z+\pi} dz$$

$$= -\int_{\pi}^{\infty} \frac{dz}{z+\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n}$$

$$= -\sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{dz}{z+\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n}.$$

For values of z between $(2k-1)\pi$ and $(2k+1)\pi$ the sum $\sum_{n=1}^{\infty}(-1)^{n+1} \sin nz/n$ becomes $z/2 - k\pi$. We have then:[1]

$$A = -\sum_{k=1}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{dz}{z+\pi} \left(\frac{z}{2} - k\pi \right)$$

$$= -\frac{1}{2} \sum_{k=1}^{\infty} \left[z - (2k+1)\pi \log (z+\pi) \right]_{(2k-1)\pi}^{(2k+1)\pi}$$

$$= \frac{\pi}{2} \sum_{k=1}^{\infty} \left[(2k+1) \log (k+1) - (2k+1) \log k - 2 \right] = \frac{\pi}{2} \lim_{r\to\infty} u_r,$$

where

$$u_r = \sum_{k=1}^{r} \left[(2k+1) \log (k+1) - (2k+1) \log k - 2 \right]$$

$$= (2r+1) \log (r+1) - 2 \sum_{1}^{r} \log k - 2r.$$

Applying Stirling's theorem, we have:

$$\lim_{r\to\infty} e^{u_r} = \lim_{r\to\infty} \frac{(r+1)^{2r+1}}{e^{2r}(r!)^2} = \lim_{r\to\infty} \frac{(r+1)^{2r+1}}{e^{2r}} \cdot \frac{e^{2r}}{2\pi r^{2r}} = \lim_{r\to\infty} \frac{1}{2\pi} \left(1 + \frac{1}{r} \right)^{2r+1} = \frac{e^2}{2\pi},$$

whence $A = \pi - \frac{1}{2}\pi \log 2\pi$.

The second sum may similarly be written

$$B = -\int_{0}^{\infty} \frac{dz}{z+\pi} \sum_{k=1}^{\infty} \frac{(-1)^{n+1} \sin nz}{n} = \pi - \frac{\pi}{2} \log 2\pi - \int_{0}^{\pi} \frac{dz}{z+\pi} \cdot \frac{z}{2},$$

which easily reduces to $\frac{1}{2}\pi - \frac{1}{2}\pi \log \pi$.

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[1] Chrystal, *Algebra*, II, 308.

3344 [1928, 446]. *Proposed by B. F. Yanney, Wooster, Ohio.*

Two altitudes of a given triangle and the side from whose extremities these altitudes are drawn meet in collinear points the corresponding sides of the orthic triangle of the given triangle.

Solution by H. A. Do Bell, New York State College for Teachers.

In the given triangle, ABC, let the altitudes from A, B, C intersect the opposite sides in D, E, F, respectively, and denote the orthocenter by H. It will be shown first that the altitudes AD, BE, and the side AB meet the corresponding sides of the orthic triangle in collinear points. The procedure is to select two triangles which are perspective from a center, thereby satisfying Desargues's Theorem. The triangles DEF and BAH are perspective from C. Therefore, if we set $(DE)\,(BA) = X$; $(DF)\,(BH) = (DF)\,(BE) = Y$; $(EF)\,(AH) = (EF)\,(AD) = Z$; the points X, Y, Z are collinear.

The other two cases of combining two of the three altitudes are shown similarly using centers A and B.

The property of perpendicularity which the altitudes possess plays no role in the above solution, but the fact that they are concurrent at H is all important. Hence, similar theorems can be formulated by using the medians (meeting in the centroid) and also by using proper angle-bisectors (meeting in the in-center and three ex-centers).

Also solved by Rufus Crane, E. D. Eaves, L. W. Johnson, Harry Langman, J. H. Neelley, and A. Pelletier.

3345 [1928, 446]. *Proposed by B. F. Finkel, Drury College.*

A mill wheel of radius a revolves so that its rim has a velocity v, and drops of water are thrown off from the rim. Find the envelope of the paths of the drops.

Solution by P. S. Dwyer, Antioch College.

Assume the origin to be at the center of the wheel, that the rotation is positive, and that the angular displacement of a point on the wheel is α. We then have the parametric equations,

$$x = a\cos\alpha - vt\sin\alpha, \quad y = a\sin\alpha + vt\cos\alpha - \tfrac{1}{2}gt^2,$$

which locate at any time, t, the position of a drop starting from any point $(a\cos\alpha, a\sin\alpha)$ on the circumference of the wheel. If we eliminate the time parameter from these equations we get

$$y = a\csc\alpha - x\cot\alpha - \tfrac{1}{2}gv^{-2}\csc^2\alpha(a\cos\alpha - x)^2, \tag{1}$$

which is the family of parabolas traversed by the various drops.

Differentiating with respect to α, we find

$$0 = v^{-2}\csc^2\alpha(a\cos\alpha - x)(v^2 - ga\csc\alpha + gx\cot\alpha). \tag{2}$$

(*a*) When $a\cos\alpha - x = 0$, $y = a\sin\alpha$, and we have the equation of the wheel's circumference, $x^2 + y^2 = a^2$.

(*b*) $v^{-2}csc^2\alpha = 0$ is inapplicable.

(*c*) The last factor gives

$$x = a\sec\alpha - v^2g^{-1}\tan\alpha. \tag{3}$$

Substituting (3) in (1) and simplifying, we get

$$y = \tfrac{1}{2}v^2g^{-1} - \tfrac{1}{2}gv^{-2}\left[(v^4g^{-2} + a^2)\tan^2\alpha - 2av^2g^{-1}\sec\alpha\tan\alpha\right]. \tag{4}$$

Squaring x in equation (3), we find

$$x^2 - a^2 = (v^4g^{-2} + a^2)\tan^2\alpha - 2av^2g^{-1}\sec\alpha\tan\alpha\ ; \tag{5}$$

and eliminating α from (4) and (5), we find as the equation of the envelope:

$$x^2 = -2v^2g^{-1}\left[y - \tfrac{1}{2}gv^{-2}(v^4g^{-2} + a^2)\right].$$

The envelope is a parabola having its vertex at a distance of $\frac{1}{2}g(v^2g^{-2}+a^2v^{-2})$ above the center of the wheel.

Also solved by W. B. Campbell, William Hoover, Enrique Linares, and Paul Wernicke.

3346 [1928, 446]. *Proposed by Frank Irwin, University of California.*

Show that, if $P_1P_2\cdots P_nP_1$ be any polygon of n sides, the broken line whose segments are parallel and equal to P_1P_k, P_2P_{k+1}, $\cdots$, P_nP_{k-1} $(P_{n+i}=P_i)$ in order will close. Generalize.

Again let S_i be the middle point of P_iP_{i+1}, $i=1, 2, \cdots, n$, $(P_{n+1}=P_1)$; show that the broken line whose sides are parallel and equal to P_1S_k, P_2S_{k+1}, $\cdots$, P_nS_{k-1} in order will close. Generalize.

Solution by the Proposer.

The propositions are quite obviously true when once stated, so that their only interest is in the propositions themselves. They hold in three, or n, dimensions.

Let α_i be the vector from any origin, O, to P_i. We shall generalise our first proposition as follows: *Let $Q_1, Q_2, \cdots, Q_n$ be the n points P_i in any order, and similarly $R_1, R_2, \cdots, R_n$. Then the broken line formed with the vectors Q_1R_1, Q_2R_2, $\cdots$, Q_nR_n will close*; for since the vector $P_iP_j = \alpha_j - \alpha_i$, we see that in the vector sum, $Q_1R_1 + Q_2R_2 + \cdots$, each α_i will occur once with a plus sign and once with a minus sign and the vector sum will be zero.

To generalise the second proposition, *let S_i lie on P_iP_{i+1} and let $P_iS_i/P_iP_{i+1} = l$, $i=1, 2, \cdots, n$. Let $Q_1, Q_2, \cdots, Q_n$ be, as before, the points P_i in any order, and let $R_1, R_2, \cdots, R_n$ now be the points S_i in any order. Then again the broken line formed with the vectors, Q_1R_1, Q_2R_2, $\cdots$, will close*; for the vector $OS_i = (1-l)\alpha_i + l\alpha_{i+1}$, so that the vector sum, $Q_1R_1 + Q_2R_2, \cdots$, will equal

$$(1-l)(\alpha_1 + \alpha_2 + \cdots) + l(\alpha_1 + \alpha_2 + \cdots) - (\alpha_1 + \alpha_2 + \cdots) = 0.$$

A familiar case of this proposition is that a triangle may be constructed with sides equal and parallel to the medians of any triangle.

We may generalize further by using in place of the points Q_i (in the last proposition) points T_i, where T_i lies on P_iP_{i+1} and $P_iT_i/P_iP_{i+1}=m$. The last proposition is the special case $m=0$. Further generalisations readily suggest themselves.

Also solved by Harry Langman, Enrique Linares, A. Pelletier, J. Rosenbaum, and Paul Wernicke.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Fine Memorial Mathematics Hall, which will be erected at Princeton University at a cost of $400,000 in memory of the late Henry B. Fine, for many years a professor of mathematics and dean of science, will be started in the near future.

The New York Academy of Sciences offers its A. Cressy Morrison prize "No. 1" for 1930 for a paper on solar and stellar energy. Compteition is open to all.

At Hunter College of the City of New York, exceptional lecture courses on mathematical statistics will be given in the evening session during 1929–30. The department of mathematics has been fortunate in securing the part time services of Mr. Arne Fisher, well-known statistician of the Western Union Company. Mr. Fisher has agreed to offer, in conjunction with the regular curricular work, two such courses.

Assistant Professor E. C. Bower, of the department of astronomy and mathematics at Ohio Wesleyan University, has been appointed Martin Kellogg fellow at the Lick Observatory of the University of California, with academic residence at Berkeley.

At Ohio State University, Professor Alfred Lande, of the University of Tübingen, has been appointed visiting professor of theoretical physics for the autumn and winter quarters of 1929–30. Dr. L. H. Thomas, of Trinity College, Cambridge, has been appointed visiting assistant professor of theoretical physics for the autumn, winter, and spring quarters.

Dr. W. L. Ayres has been appointed to an assistant professorship at the University of Michigan.

Professor D. P. Bartlett, of the Massachusetts Institute of Technology, has retired.

Professor R. D. Carmichael has been appointed administrative head of the department of mathematics at the University of Illinois, as successor to Professor E. J. Townsend, who has retired.

Assistant Professor P. D. Edwards, of Ball Teachers College, has been promoted to an associate professorship.

Mr. W. I. Foster, of Rochester Junior College, has been appointed assistant professor of mathematics at Northern Montana School.

Dr. B. F. Kimball, of Cornell University, has been appointed to an assistant professorship at the University of New Hampshire.

Assistant Professor E. E. Libman, of the University of Illinois, has resigned to accept a position in the marine and air craft engineering department of the General Electric Company.

Associate Professor T. A. Pierce, of the University of Nebraska, has been promoted to a professorship.

Assistant Professor S. A. Schelkunoff, of the State College of Washington, has resigned to accept a position in the Bell Telephone Laboratories, New York City.

Mr. C. K. Sherer, of the University of Nebraska, has been appointed head of the department of mathematics at Texas Christian University.

Assistant Professor G. W. Smith, of the University of Kansas, has been promoted to an associate professorship.

Dr. Roxana H. Vivian has been appointed professor of mathematics at Hartwick College, Oneonta.

Mr. E. H. Wells, of Princeton University, has been appointed to an assistant professorship at the University of New Hampshire.

Dr. Edgar W. Woolard, of the George Washington University, has been promoted to an assistant professorship of mathematics.

Dr. Harvey A. Zinszer, professor of physics and acting professor of mathematics at Hanover College, has been appointed professor of physics and astronomy at the Kansas State Teachers College.

The following appointments to instructorships are announced:

Connecticut College, Miss Grace Shover.

Newark College of Engineering, Mr. J. H. Fithian.

New York University, Dr. D. A. Flanders, Mr. E. H. Johnson, Mr. A. S. Peters.

Ohio State University, Dr. P. M. Swingle.

Pennsylvania State College, Dr. Leo Zippin.

Professor F. C. Kent, of the Oregon State Agricultural College, died June 11, 1929.

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[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the Editor-in-Chief, W. H. Bussey, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. Johnson, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the Secretary Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-27, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1929.

Illinois, Carthage, Ill., May 3-4.
Indiana, Culver Military Academy, May 3-4.
Iowa, Fairfield, Iowa, April 26-27.
Kansas, Topeka, Kansas, February 2.
Kentucky, Lexington, Ky., April 13.
Louisiana-Mississippi, Lafayette, La., April 12-13.
Maryland-District of Columbia-Virginia, George Washington University, May 4.
Michigan, Ann Arbor, Mich., March 16.
Minnesota, St. Paul, Minn., May 11.
Missouri, Kansas City, Mo., November 16
Nebraska.
Ohio, Columbus, Ohio, April 4.
Philadelphia, University of Pennsylvania, November 30.
Rocky Mountain, Greeley, Colo., April 12-13.
Southeastern, Macon, Ga., April 19-20.
Southern California, University of Redlands, March 9.
Texas, Houston, Texas, Jan. 26.

Affiliated Organizations: The New England Association of Teachers of Mathematics. The National Council of Teachers of Mathematics.

THE THIRTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The thirteenth summer meeting of the Mathematical Association of America was held, by invitation, at the University of Colorado, Boulder, Colo., on Monday and Tuesday, August 26–27, 1929, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Two hundred thirty-nine were present at the meetings, including the following one hundred twenty-one members of the Association.

E. F. ALLEN, Bryn Athyn, Pa.

CLARA L. BACON, Goucher College
W. D. BATEN, Ann Arbor, Mich.
SUZAN R. BENEDICT, Smith College
W. J. BERRY, George Washington University
R. W. BRINK, University of Minnesota
JACK BRITTON, University of Colorado
MYRTLE C. BROWN, Teachers College, Denton, Tex.
R. E. BRUCE, Boston University
H. E. BUCHANAN, Tulane University
J. E. BURNAM, Simmons College
W. H. BUSSEY, University of Minnesota

W. D. CAIRNS, Oberlin College
C. C. CAMP, University of Nebraska
E. W. CHITTENDEN, University of Iowa
A. G. CLARK, Colorado Agricultural College
ABRAHAM COHEN, Johns Hopkins University
TERESA COHEN, Pennsylvania State College
E. E. COLYER, Hays (Kans.) State Teachers College
LENNIE P. COPELAND, Wellesley College
D. R. CURTISS, Northwestern University

OSCAR DAHLENE, University of Alabama
D. R. DAVIS, University of Oregon
H. T. DAVIS, Indiana University
ALICE C. DEAN, Rice Institute
W. W. DENTON, University of Michigan
L. L. DINES, University of Saskatchewan
E. L. DODD, University of Texas

C. M. ERIKSON, Michigan State Normal College
J. R. EVERETT, Colorado School of Mines

J. C. FITTERER, Colorado School of Mines
L. R. FORD, Rice Institute
W. B. FORD, University of Michigan
T. C. FRY, Bell Telephone Laboratories, New York, N. Y.

M. G. GABA, University of Nebraska
J. L. GIBSON, University of Utah
CORNELIUS GOUWENS, Iowa State College
F. L. GRIFFIN, Reed College
LOIS W. GRIFFITHS, Northwestern University

SIDNEY HACKER, University of Colorado
LAURENCE HAMPTON, University of Oklahoma
J. A. HARDIN, Centenary College
W. L. HART, University of Minnesota
E. R. HEDRICK, University of California at Los Angeles
CORA B. HENNEL, Indiana University
GERTRUDE A. HERR, Iowa State College
H. C. HICKS, Texas Tech. College
T. H. HILDEBRANDT, University of Michigan
GOLDIE P. HORTON, University of Texas
JEWELL C. HUGHES, University of Arkansas
J. W. HURST, Montana State College
C. A. HUTCHINSON, University of Colorado

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota

A. J. KEMPNER, University of Colorado
CLARIBEL KENDALL, University of Colorado
J. H. KINDLE, University of Cincinnati
H. W. KUHN, Ohio State University

C. G. LATIMER, University of Kentucky
D. N. LEHMER, University of California at Berkeley
O. C. LESTER, University of Colorado
A. J. LEWIS, University of Denver
C. F. LEWIS, Kansas State Agric. College
G. H. LIGHT, University of Colorado
W. V. LOVITT, Colorado College
R. G. LUBBEN, University of Texas

S. L. MACDONALD, Colorado State Agric. College
ELSIE MCFARLAND, University of Oklahoma
A. S. MCMASTER, University of Colorado
G. E. MARCH, South Dakota State School of Mines
E. L. MICKELSON, New Mexico State Teachers College
R. R. MIDDLEMISS, University of Colorado
W. E. MILNE, University of Oregon
U. G. MITCHELL, University of Kansas
THIRZA A. MOSSMAN, Kansas State Agric. College
R. L. MOORE, University of Texas
C. A. MURRAY, West Texas State Teachers College

W. L. NELSON, University of Colorado
C. V. NEWSON, University of New Mexico
J. A. NYSWANDER, University of Michigan

H. L. OLSON. Michigan State College